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METHODS OF DETERMINING NATURAL MODES AND FREQUENCIES FOR TRANSVERSE VIBRATIONS OF BEAMS

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METHODS OF DETERMINING NATURAL MODES AND FREQUENCIES OF TRANSVERSE VIBRATIONS OF BEAMS

CHAPTER I

INTRODUCTION

Much effort has been spent in finding exact solutions of the differential equations of vibration problems. In only the simplest of structures, viz. the uniform beam, is it possible to determine rigorous solutions that include the effects of rotary inertia and shear resistance of the beam. For beams whose cross section varies in a simple manner along the axis of the beam it is possible, though somewhat difficult to obtain exact solutions that include the effects of rotary inertia. For arbitrary variation of the mass-density distribution and arbitrarily changing elastic properties along the axis, even if they are expressible in analytical form, it has not been found possible to obtain exact solutions. Consequently the emphasis of this report lies in obtaining approximate analytical methods rather than in working out the solutions of the differential equations of a class of beams with, say, cross sections that vary according to simple analytical functions along the axis of the beam. The differential equation of the problem will be developed and it will be seen that it does not lend itself readily to the determination of approximate numerical solutions. It is much more advantageous to set up the problem in its integral equation representation, because integral equations can easily be solved by approximate methods. The reason lies in the entirely different approach that leads to the integral equations in contrast with the differential equation methods. The differential equation considers

the elastic structure in its elements: it studies the motions of a particle of the aggregate that makes up the structure and builds up - by integration - the motions of the structure as a whole. These must be compatible with the conditions of support, and it is here that the real difficulty and laboriousness of the differential equation procedure enters: the solution of the boundary-value problem, or the evaluation of the constants of integration. The integral equation considers the elastic body as a whole and it has, so to speak, all the boundary conditions of the problem written into it. It is true, that it builds up the motions of the body from "elements", but these are not those of isolated particles but simple motions of the whole structure. The integral equations are not the only ones that consider the body as a whole, for the variational methods, better known to the engineer as energy methods, also formulate the problem for the body as an entity, and it is convenient to say that mechanical problems of vibrations can be stated in either differential form or in integral form. Here, attention is focused on the integral formulations.

A few comments have to be made about the term "exact solution" that has been mentioned above, i.e. it has to be stated under what set of conditions an analytical solution shall be called exact. The equations of the problem shall be linear. This limits the magnitude of vibrations to so-called small vibrations and it requires linear stress-strain relationship. Furthermore the simplified beam theory shall be valid. This means that the Bernoulli-Navier hypothesis - plane sections which are normal to the undeformed elastic axis remain plane and normal to the deformed elastic axis - is valid, and thus any exact solution is not rigorous in the sense of the mathematical theory of

elasticity, as the warping of the cross sections has been neglected. (See Chapter III also.)

The following section demonstrates a few mathematical principles that will frequently be drawn upon in the ensuing analysis.

CHAPTER II

MATHEMATICAL PREREQUISITE

1. Notations, Matrix Calculus

Matrix calculus will be used extensively in the following sections. Thus it is necessary to list the notations that are used in this paper. They are in accord with continental practice, offer the advantage of simplicity and economy of print, and are closely related with indicial notation.

In indicial notation - as it is used hereafter - the convention is established that, if an index occurs more than once in a product, a summation over the index is to be carried out over a number of integers ranging from one to n, unless otherwise specified.

For example:

$$a_i b_i = \sum_i a_i b_i \quad , \quad a_i b_{ij} = \sum_i a_i b_{ij} \quad , \quad a_{ij} b_{jk} = \sum_j a_{ij} b_{jk}$$

$$a_{ij} b_{je} c_{ek} = \sum_j \sum_e a_{ij} b_{je} c_{ek} = \sum_j a_{ij} \left(\sum_e b_{je} c_{ek} \right) =$$

$$= a_{i1} \left(b_{11} c_{1k} + \cdots + b_{1n_4} c_{n_4 k} \right) + a_{i2} \left(b_{21} c_{1k} + \cdots + b_{2n_4} c_{n_4 k} \right) +$$

$$+ \cdots + a_{in_2} \left(b_{n_2 1} c_{1k} + \cdots + b_{n_2 n_4} c_{n_4 k} \right) . \qquad (1)$$

The ranges of the indices i, j, k, l shall be n_1, n_2, n_3, n_4, respectively.

Matrices are designated by arabic capitals; vectors, that is, matrices with either one row or one column only, by lower case arabic letters. Row-vectors have superscripts, column-vectors have subscripts. Bold face letters (or underlined letters) shall be used where confusion of matrices with their elements is possible. The following are equivalent notations of

an m-row n-column matrix, or an n by n matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \quad \begin{bmatrix} i=1,\ldots,m \\ j=1,\ldots,n \end{bmatrix} \quad = \quad \mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{pmatrix} = \underline{\mathbf{A}} \quad ;$$

$$\underline{\mathbf{a}}_r = \mathbf{a}_r = \begin{pmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{mr} \end{pmatrix} \quad ; \qquad \underline{\mathbf{a}}^s = \mathbf{a}^s = (a_{s1}, a_{s2}, \ldots, a_{sn})$$

$$\text{or also: } \mathbf{a}_r = \{ a_{1r}, a_{2r}, \ldots, a_{mr} \} \; . \tag{2}$$

Multiplication of a matrix by a scalar, - as multiplication of every element of the matrix by the scalar;

$$\lambda \underline{\mathbf{A}} = (\lambda a_{ij}) \tag{3}$$

Addition and subtraction of matrices, - as algebraic addition of corresponding elements of the matrices. (Corresponding means situated at equal places, having the same address or coordinates)

$$\mathbf{A} - \mathbf{B} + \mathbf{C} = \mathbf{D} = (d_{ij}), \text{ where } d_{ij} = a_{ij} - b_{ij} + c_{ij} \; . \tag{4}$$

Addition and subtraction are associative and distributive but are, of course, restricted to aggregates of matrices which have all the same number of rows and the same number of columns.

Multiplication of a vector by another vector, - the familiar scalar product of vector algebra is written as;

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}' \mathbf{b} = (a_1, a_2, \ldots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \equiv (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) = a_i b_i \; , \tag{5}$$

and with it, the multiplication of matrices by vectors and matrices are easily defined. The ' indicates transpositions of rows and columns. (The dyadic product of vectors is not needed here.)

Multiplication of a matrix by a column-vector, - is a new column vector whose i-th component is the scalar product of the i-th row vector of the matrix with the column vector:

$$A\,b = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{pmatrix} b = \begin{pmatrix} a^1 b \\ a^2 b \\ \vdots \\ a^m b \end{pmatrix}; \text{ with } \quad a^r b = a_{ri}\, b_i \,. \tag{6}$$

Multiplication of two matrices, - is a new matrix whose ij-th element (that is, the i-th component of the j-th column vector) is defined as the scalar product of the i-th row vector of the first factor-matrix into the j-th column vector of the second factor-matrix. The two matrices must be compatible, i.e. the number of the elements in the rows of the first matrix must be equal to that of the elements in the columns of the second. Symbolically the product is defined as;

$$C \equiv A\,B = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{pmatrix} \left(b_1\, b_2 \cdots b_n \right) = \begin{pmatrix} a^1 b_1 & a^1 b_2 & \cdots & a^1 b_n \\ a^2 b_1 & a^2 b_2 & \cdots & a^2 b_n \\ \vdots & \vdots & \cdots & \vdots \\ a^m b_1 & a^m b_2 & \cdots & a^m b_n \end{pmatrix} = (a_{ik}\, b_{kj}) \quad \begin{bmatrix} i=1,\ldots,m \\ k=1,\ldots,s \end{bmatrix} \tag{7}$$

The matrix product is associative, distributive but not, in general, commutative. The following relations are true;

$$\begin{aligned} (A\,B)\,C &= A\,(B\,C) \\ (A+B)\,C &= A\,C + B\,C \\ A\,B &\neq B\,A \,, \end{aligned} \tag{8}$$

Multiplication of a matrix with special matrices, viz. unit-matrix, and diagonal matrices. The leading (or main) diagonal of a square matrix is the line which contains the elements with two equal subscripts, (a_{ii}). It is only defined for square matrices. A matrix whose elements are all zero is called the zero-matrix. If at least one of the elements of the leading diagonal of a square matrix does not vanish, while all others are zero, then it is a diagonal-matrix. A diagonal matrix with elements all equal to unity is called unit-matrix. A diagonal matrix with elements all equal to a scalar is called a scalar-matrix.

The unit matrix, $$E=(e_1,\cdots,e_n)=\begin{pmatrix}e^1\\ \vdots\\ e^n\end{pmatrix}=(\delta_{ij})=\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\ \cdots&\cdots&\cdots&\cdots\\0&0&\cdots&1\end{pmatrix}$$

has the property, $$AE=EA=A. \tag{9}$$

The diagonal matrix, $$D=\begin{pmatrix}d_1&0&\cdots&0\\0&d_2&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&d_n\end{pmatrix}$$

can either premultiply or postmultiply a matrix A, yielding DA and AD respectively. Premultiplication by a diagonal matrix multiplies the rows of the second factor -, postmultiplication by a diagonal matrix multiplies the rows of the first factor - by the corresponding diagonal element, viz.:

$$DA=\begin{pmatrix}d_1a_{11}&d_1a_{21}&\cdots&d_1a_{1n}\\d_2a_{21}&d_2a_{22}&\cdots&d_2a_{1n}\\d_3a_{31}&d_3a_{32}&\cdots&d_3a_{1n}\\ \vdots&\vdots&\vdots&\vdots\\d_m a_{m1}&d_m a_{m2}&\cdots&d_m a_{mn}\end{pmatrix};\quad AD=\begin{pmatrix}d_1a_{11}&d_2a_{12}&d_3a_{13}&\cdots&d_n a_{1n}\\d_1a_{11}&d_2a_{22}&d_3a_{23}&\cdots&d_n a_{2n}\\ \vdots&\vdots&\vdots&\vdots&\vdots\\d_1a_{m1}&d_2a_{m2}&d_3a_{m3}&\cdots&d_n a_{mn}\end{pmatrix}. \tag{10}$$

Repeated products of matrices with the same factor are written as powers of that matrix. The zero-th power of a matrix is defined to be the unit matrix:

$$A^0 = E\ ;\quad A^1 = A\ ;\quad A^2 = A\cdot A\ ;\ \cdots,\quad A^p \cdot A^q = A^{p+q}\ . \tag{11}$$

Operations on matrices that are frequently used are defined next.

Transposition, - is an interchange of rows and columns and designated by a prime '. Thus $(A')' = A$. The transpose of a product is equal to the product of the transposed factors, in reversed order,

$$(A\,B)' = B'A' \tag{12}$$

for, if $C = AB = (a_{ik}\, b_{kj}) = (c_{ij})$; then

$$C' = (c_{ij})' = (c_{ji}) = (a_{jk}\, b_{ki}) = (b_{ik})'(a_{kj})' = B'A'\ .$$

The inverse (or reciprocal) of a matrix, - is defined as that matrix whose premultiplication, or postmultiplication, with A gives the unit matrix E , viz.:

$$A^{-1}A = A\,A^{-1} = E\ . \tag{13}$$

The theory of determinants shows that the ij-th element of the inverse matrix is equal to

$$(a_{ij})^{-1} = \frac{c.f.\ a_{ji}}{|A|} \tag{14}$$

where $c.f.\ a_{ji}$ is the cofactor of the element a_{ji} , and $\det A = \|A\|$ the determinant of the matrix A .

More details may be found in any text-book on linear algebra. An easy exposition is W. L. Ferrar's, Algebra, Claredon Press 1941, and a very clear and

more advanced treatise is R. Zurmühl's, Matrizen, Springer 1950, (in German)

The inverse of a product is equal to the product of the inverse of of the factors in reversed order:

$$(AB)^{-1} = B^{-1}A^{-1}; \tag{15}$$

for then only: $(AB)(AB)^{-1} = ABB^{-1}A^{-1} = AEA^{-1} = AA^{-1} = E.$

Frequently multiple matrix products will be encountered which originate from multiple summations, and the problem of writing such summations as matrix products arises. Thus a few typical matrix products will be written in summation form. Only the ij-th element is listed. The ranges of i and j are one to m and one to n, respectively.

1. Example: the simple summation $s_{ij} = a_{ik} b_{jk} = \sum_r a_{ik} b_{jk}$ is given. If the order of the indices in the second factor were to be reversed this summation would coincide with the definition of the matrix product AB. As a simultaneous reversal of the indices and transposition leave the second matrix unchanged, $b_{jk} = (b_{kj})'$, the following is true:

$$S = (s_{ij}) = (a_{ik} b_{jk}) = (a_{ik} (b_{kj})') = AB'. \tag{16}$$

If the individual products $a_{ik} b_{jk}$ in the summation $\sum^k a_{ik} b_{kj}$ are multiplied, or weighted, by factors, d_k, that is, if the summation $s_{ij} = \sum^k a_{ik} b_{kj} d_k$ is given, observe that the weighting factor (or numerical integration factor) carries the column index k of the factor a_{ik}. This means that every column of the matrix A is multiplied by a certain factor

d_K . The second of the equations (10) on page 7 shows that such a matrix is the product of a matrix $\mathbb{A}$, postmultiplied by the diagonal matrix $\mathbb{D}=\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_K \end{pmatrix}$ Thus one obtains:

$$S=(s_{ij}) = (a_{iK}\, b_{Kj}\, d_K) = \left((a_{iK}\, d_K)(b_{Kj})'\right) = A\,D\,B' , \tag{17}$$

which is also equal to, $S = A(BD)'$; since $D' = D$.

2. Example: given the matrix S whose ij-th element is defined by the double sum: $s_{ij} = a_{ih}\, b_{hK}\, c_{Kj}$. The object is to write S as a matrix product. If one sets $p_{iK} = a_{ih}\, b_{hK}$, the problem is reduced to that of the first example, for then, $s_{ij} = a_{ih}\, b_{hK}\, c_{Kj} = p_{iK}\, c_{Kj}$. This is the ij-th element of the product PC , where $P=AB$, and, since the matrix product is associative, there follows:

$$S = (s_{ij}) = (a_{ih}\, b_{hK}\, c_{Kj}) = (a_{ih})(b_{hK})(c_{Kj}) = ABC .$$

If the two summations, over h and k, are weighted with two sets of factors, say, $N=(n_{hh})$ and $M=(m_{KK})$, a double summation of the following type is obtained, $s_{ij} = a_{ih}\, b_{hK}\, c_{Kj}\, n_h\, m_K$, and it can (using the result of the previous example) be written as the ij-th element of the matrix product listed below:

$$S = (s_{ij}) = ANBMC ; \quad N = (n_{hh}) = \begin{pmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_h \end{pmatrix} , \quad M = (m_{KK}) = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_K \end{pmatrix} .$$

If the order of the indices does not agree with the one listed above, transpositions must be made. Not all summations can, of course, be written as typical elements of matrix products: the indices of the factors

must be such that they can be arranged so that the first index of a factor is the same as the last index of the preceeding factor. The number of possible weighting factors is the same as the number of summations. So, for instance, the quintuple sum: $S_{ij} = a_{ik} b_{kl} c_{lm} d_{mn} e_{nr} f_{rj} m_k n_l o_m p_n q_r$, where m_k, n_l, o_m, p_n, q_r are weighting factors, can be written as the ij-th element of the matrix product: $S = (S_{ij}) = AMBNCODEQF$. Summations, as they have been illustrated in the two examples, will be encountered when certain integrals are evaluated numerically.

2. Series Expansions of Arbitrary Functions

Another concept of greatest importance and frequent application is the problem of the series expansion of arbitrary functions. It will be reviewed briefly, in somewhat more general form than it is found in the usual texts of methematical physics, such as H. & B.S. Jeffreys, Mathematical Physics, Cambridge University Press 1950, or R. Courant & Hilbert, Methoden der mathematischen Physik, Interscience, New York.

Formulation of the problem:

Given a set of non-overlapping subintervals of the line, i.e. intervals of the form: $a_i \leq x \leq b_i$. In this are defined a piecewise continuous function f(x) and a set of functions, designated by $\{\varphi(x)\}$. All the functions are subject to the following restrictions:

(1) They must be integrable square, i.e. their squares must be integrable in the interval.

(2) All inner products of the functions must exist in the interval.

Recall that the inner product of two functions g(x) and h(x) over the interval $a \leq x \leq b$ is defined as:

$$\int_a^b g(x)\,h[x]\,dx \equiv (g,h) = (h,g) .$$

The inner product of a function with itself is called the norm of the funetion and written as:

$$N\bar{g} = (g,g) \equiv \int_a^b g^2[x]\,dx .$$

Introduce the following abbreviations:

$$\left.\begin{aligned} (\varphi_\nu, \varphi_\mu) &= b_{\nu\mu} = b_{\mu\nu} \\ (f, \varphi_\mu) &= c_\mu \end{aligned}\right\} \quad \text{the } \mu\text{-component of f in the } \varphi\text{'s.} \tag{18}$$

It is desired to approximate f(x) by a linear aggregate of the φ's,

$$f \approx \bar{f}[x] \equiv a_\nu \varphi_\nu [x] . \tag{19}$$

in such a way that the approximation is best in the least squared error sense, i.e. that the integral over the square of the error is a minimum, viz.

$$M[x] = \int_a^b (f-\bar{f})^2 dx = N(f-\bar{f}) = (f-\bar{f}, f-\bar{f}) = \text{minimum.} \tag{20}$$

The problem consists now in determining the expansion coefficients a_μ in such a way that M(x) takes on its minimum value. This implies that all the partial derivatives of M with respect to all a_μ must be zero:

$$\partial M / \partial a_\mu = 0 , \qquad \text{for all } \mu = 1,2, \ldots n. \tag{20'}$$

This set of n linear non-homogeneous equations determines the a_μ.

Carrying out the partial differentiation:

$$\partial M/\partial a_\mu = (\partial/\partial a_\mu)\int [f - a_\nu \varphi_\nu]^2 dx = 2\int [f - a_\nu \varphi_\nu] \frac{\partial [f - a_\nu \varphi_\nu]}{\partial a_\mu} dx ,$$

and noticing that $\partial [f - a_\nu \varphi_\nu]/\partial a_\mu = -\varphi_\mu$, there follows:

$\frac{\partial M}{\partial a_\mu} = -2 \int [f - a_\nu \varphi_\nu] \varphi_\mu \, dx = 0$. Cancelling the factor -2 and performing the multiplication in the integrand, one obtains:

$\int f \varphi_\mu dx - a_\nu \int \varphi_\nu \varphi_\mu \, dx = 0$, (as a_ν is a constant factor)

which can also be written more conveniently as an equality of inner products:

$$(f, \varphi_\mu) = a_\nu (\varphi_\nu, \varphi_\mu) \quad . \tag{20''}$$

With the abbreviations (18) this equation can be written as:

$$\mathbf{c} = \mathbf{B}\mathbf{a} \tag{21}$$

where $\mathbf{B}$ is the matrix of the $b_{\nu\mu}$, and $\mathbf{c}$ and $\mathbf{a}$ are the column vectors:

$$\mathbf{c} = \{c_1, c_2, \cdots, c_n\} \quad ; \quad \mathbf{a} = \{a_1, a_2, \cdots, a_n\} \quad .$$

Equation (21) has the solution: $\mathbf{a} = \mathbf{B}^{-1}\mathbf{c}$. (22)

The ij-th element of the inverse matrix is given by: $(b_{ij})^{-1} = c.f. \, b_{ji} / \|\mathbf{B}\|$

By virtue of the imposed restrictions, $\mathbf{c}$ exists (i.e. is finite), and thus $\underline{\mathbf{a}}$ exists if, and only if, $\mathbf{B}^{-1}$ exists. This implies that the determinant of the matrix $\mathbf{B}$ must not vanish, viz.

$$\|\mathbf{B}\| \neq 0 \quad . \tag{23}$$

If one writes this in expanded form, and introduces the definition of b_{ij}, one arrives at the fundamental equation, the so-called Gramian,:

$$\Gamma(\varphi_1, \varphi_2, \cdots, \varphi_n) = \begin{vmatrix} (\varphi_1, \varphi_1) & \cdots & (\varphi_1, \varphi_n) \\ \vdots & & \vdots \\ \varphi_n, \varphi_1 & \cdots & (\varphi_n, \varphi_n) \end{vmatrix} \neq 0 \tag{23'}$$

which the functions $\{\varphi\}$ must satisfy, in order that the expansion (19) is possible.

The non-vanishing of the Gramian implies that the functions are linearly independent, for, if this were not so, a relation

$$d_1 \varphi_1 + d_2 \varphi_2 + \cdots + d_n \varphi_n = 0 \qquad (24)$$

with not all coefficients d_i zero would be true. To prove this, note that from equation (24) any of the φ's can be expressed as a linear combination of all others. After introducing this into the equation (23') one obtains a determinant with a column that is a linear combination of the other columns. But such a determinant is always zero, hence equation (24) cannot be satisfied.

Returning to the expansion problem, the approximation f(x) (eq. 19) becomes with eq. (22)

$$\bar{f} = a_\nu \varphi_\nu = b^{-1}_{\nu\mu} c_\mu \varphi_\nu = \varphi' B^{-1} c .$$

Obviously M(x), as a definite integral with non-negative integrand, is never less than zero. From this remark a generalized form of the Bessel inequality and the Parseval equality can easily be derived.

M was given as:

$$M = \int (f - a_\nu \varphi_\nu)^2 dx = (f - a_\nu \varphi_\nu, f - a_\nu \varphi_\nu) \geq 0 .$$

Expanding and carrying out the integrations, with the abbreviations (18), this becomes:

$$Nf - 2 a_\nu (f, \varphi_\nu) + a_\nu a_\mu (\varphi_\nu, \varphi_\mu) =$$

$$Nf - 2 a_\nu c_\nu + a_\nu a_\mu b_{\nu\mu} \geq 0,$$

and introducing a_ν from (22) as $a_\nu = b^{-1}_{\nu\mu} c_\mu$, one obtains:

$$Nf - 2 b^{-1}_{\nu\mu} c_\nu c_\mu + b^{-1}_{\nu\mu} c_\mu b^{-1}_{\nu\mu} c_\nu b_{\nu\mu} \geq 0.$$

Since $b_{\nu\mu}\, b^{-1}_{\nu\mu}$ is the product of a matrix and its inverse and thus equal to the unit matrix, one obtains finally

the generalized Bessel inequality: $$b^{-1}_{\nu\mu}\, c_\nu\, c_\mu \leq Nf \quad . \tag{25}$$

As the right-hand side of (25) is independent of the range of the summations of ν and μ from one to n, the summations on the left-hand side can be extended from one to infinity.

Now, there always exists a number $n > N$ such that the squared error M is less than an arbitrarily small ϵ, i.e.

$$M = \int [f - a_\nu \varphi_\nu]^2 dx < \epsilon \quad , \quad \text{if} \quad n > N.$$

In the limit $n \to \infty$ M becomes zero and the Bessel inequality reduces to an equality,

the generalized Parseval equality: $$b^{-1}_{\nu\mu}\, c_\nu\, c_\mu = Nf \quad . \tag{26}$$

The equation (26) can also be called a generalized completeness relation. An infinite set of lineraly independent functions is complete if it permits the approximation in the mean (least squared error sense) of any piecewise continuous function to any desired degree of accuracy.

It should be noted that the equation:

$$\lim_{n\to\infty} \int [f - a_\nu \varphi_\nu]^2 dx \quad , \text{ or } \underset{n\to\infty}{\text{l.i.m.}} \; (f - a_\nu \varphi_\nu) = 0 \tag{27}$$

(l.i.m. means limit in the mean)

does not imply the equality:

$$f = \sum_\nu^\infty a_\nu \varphi_\nu \quad . \tag{28}$$

The latter is only assured, if the sequence of the $\{a_\nu \varphi_\nu\}$ converges uniformly in the subset of the line, because only then is it permissible to move the integral sign past the summation sign.

Very important for later applications are the statements:

Lemma 1: A piecewise continuous function is uniquely determined by its expansion coefficients in a complete system of functions; or also,

Lemma 2: Two piecewise continuous functions are identical if they possess the same expansion coefficients.

The proofs follow from the fact that the difference of two such functions with the same expansion coefficients has an expansion with all coefficients equal to zero. And thus, by virtue of the generalized Parseval equality, its norm vanishes. But this implies that the difference itself is zero, hence the two functions must be identical. The expansion coefficients in a complete set describe a certain function uniquely, even if the series expansion converges only in the mean and not in the usual sense.

It has been shown so far that any set of linearly independent functions is a complete set. Among these the orthogonal (and normalized) sets occupy a preferred position, because the determination of the expansion coefficients of a function in an orthogonal system is greatly simplified, as will be shown presently.

Two functions are said to be orthogonal if their inner product vanishes, and they are normalized if their norm equals unity. Thus the set $\{\psi\}$ is orthogonal and normalized if it obeys the two equations:

orthogonality relation: $$(\psi_\mu, \psi_\nu) = \begin{cases} N\psi_\mu & \text{for } \mu = \nu \\ 0 & \text{for } \mu \neq \nu \end{cases} \qquad (29)$$

normalizing equation: $$N\psi_\nu = 1 \quad \text{for all } \nu = 1, 2, \ldots \qquad (30)$$

The equations (29) and (30) can also be united as

$$(\psi_\nu, \psi_\mu) = e_{\nu\mu} \quad ; \quad (e_{\nu\mu}) \equiv \mathbf{E} \tag{31}$$

in which case the set $\{\psi\}$ is said to be ortho-normal.

With equation (31) the matrix $\mathbf{B}$ becomes a unit matrix and thus equation (22) reduces to:

$$a = c = (f, \psi) \tag{32}$$

which determines the expansion coefficients. They are the components of f in the set of the ψ's.

The Gramian (23') is now equal to unity, and thus, orthogonal and normalized sets of functions are certainly linearly independent. (This remains valid if the orthogonal set is not normalized, for the Gramian is then equal to the product of the norms of the members of the set.) With $\mathbf{B}$ a unit matrix $\mathbf{B}^{-1}$ is also a unit matrix, and the double sums in the generalized forms of the Bessel inequality and Parseval equality become simple sums over the c's with equal subscripts:

Bessel inequality $\sum_1^N c_\nu^2 < Nf$ (33)

Parseval equality $\sum_1^\infty c_\nu^2 = Nf$ (34)

} for orthogonal functions.

With equation (32) the approximation $\bar{f}(x)$ is given by:

$$\bar{F} = c_\nu \psi_\nu = (f, \psi_\nu)\psi_\nu \,. \qquad \nu = 1, 2, \ldots \tag{35}$$

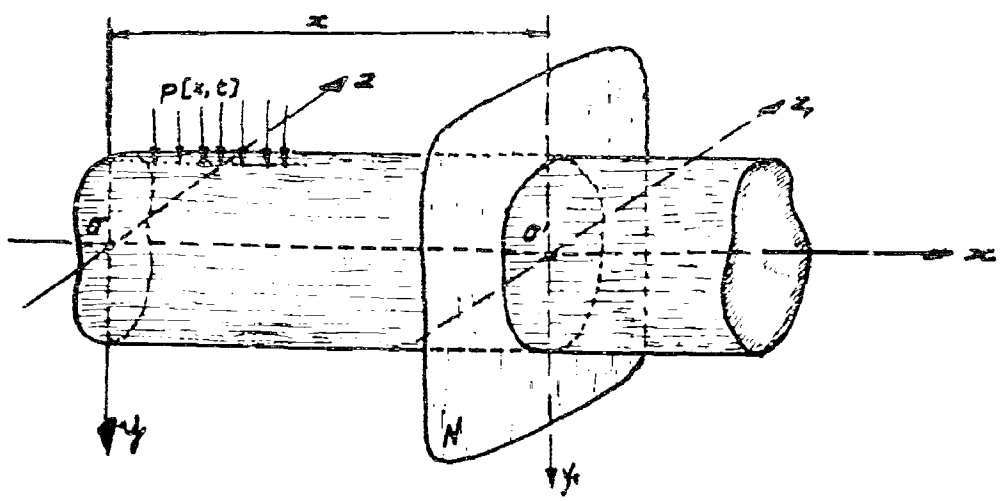
which converges in the mean.

CHAPTER III

TRANSVERSE VIBRATIONS OF STRAIGHT BARS.

THE MECHANICAL PROBLEM

1. The Differential Equation of Vibration (The Differential Approach)



Let a beam with variable cross section be defined as follows:

x-x is a straight line and N is a plane normal to it. O' is the intersection of x-x with N. The boundary curve $f_1(x) = f_1(y_1, z_1; x)$ is subject to five restrictions:

(1) O' is the centroid of the area enclosed by f_1;

(2) y_1 and z_1 are the centroidal principal axes of the area-moment of inertia of the cross section;

(3) the y_1 and z_1 axes are always parallel to the y and z axes respectively;

(4) $f_1(x)$ is a function of x only and not of time, i.e. the cross section does not change its shape during vibration. (If the cross section is hollow, rigid stiffeners must be provided to prevent change of shape.)

The fifth restriction will be formulated presently.

Suppose that some forces p(x,t) are acting on the beam in such a way that they are in the x-y plane and normal to the x-axis. If the beam was initially at rest it will then vibrate in the x-y plane only when the time dependent forces p are acting.

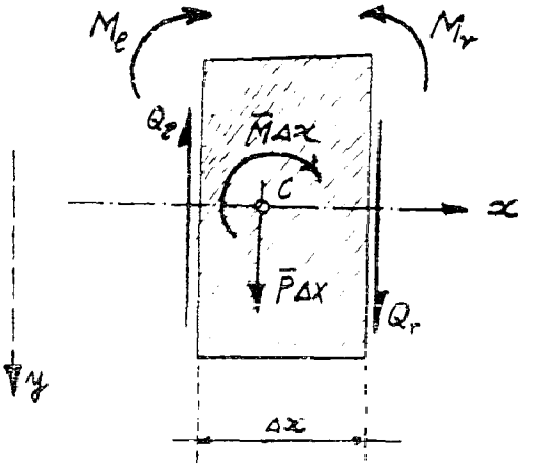
In order to study the motion of the beam, isolate a small slice which is cut out by two neighboring normal planes N and N', apply the inner forces between it and the remainder of the beam as exterior forces and formulate the equilibrium equations. These state that the sum total of all forces and moments, including the inertia forces and moments, of course, must vanish. Due to the simplifying assumptions, only two of the six equilibrium equations (in Cartesian coordinates!) are not trivial, namely, that the sum of the vertical forces have a zero-resultant and that the sum of the moments about the z-axis have a zero resultant moment vector. The following figure shows the isolated slice of length Δ x. The inner forces have been combined into a shear force Q and a bending moment M. The indices "ℓ" and "r" refer to left and right. If the beams's oscillation is in the x-y plane only, then the resultant of the shear-stress intensities of the cross section must be parallel to the y-axis. It is known, however, that this requirement can

only then be fulfilled if the exterior force acts through the shear-center of the cross section. As the forces p have been prescribed as passing through the centroid of the cross-section, it is thus necessary to specify that the shear center be in the x-y plane. It is also known that an axis of symmetry of a cross section is a locus line of the shear center. Hence the restriction which must be added is:

(5) the locus line of the shear centers of all cross sections must be in the x-y plane, i.e. in the plane of action of the exterior forces which is also the plane of vibration.

The cross section must vary gradually, otherwise the simple beam theory and the theory of the shear center are no longer applicable.

(Detailed discussion of shear centers of solid cross sections can be found in A. & L. Föppl, Drang und Zwang, vol. 2, paragraph 78, Oldenburg 1944, Munich.) Some other dynamic restrictions will be discussed in latter sections.



The figure shows the slice Δx and the resultants of the normal - and shearing stresses on the left and right faces. The resultants of the exterior and inertia forces are designated by $\bar{P}\Delta x$ and $\bar{M}\Delta x$. C is the center of gravity of the slice.

Neglecting the dead weight which is an effect of higher order, the sum of the vertical forces must be zero;

$$Q_r - Q_e + \bar{P}\Delta x = 0$$

and the sum of the moments must be zero:

$$-M_r + M_e + \Delta x(Q_r + Q_e)/2 + \bar{M}\Delta x = 0.$$

Dividing both equations by Δx and letting Δx approach zero, one obtains;

$$\left.\begin{aligned} Q' + \bar{P} &= 0, \\ -M' + Q + \bar{M} &= 0, \end{aligned}\right\}$$

where the definitions:

$$\lim_{\Delta x \to 0} \frac{M_r - M_e}{\Delta x} = M'; \quad \lim_{\Delta x \to 0} \frac{Q_r - Q_e}{\Delta x} = Q'; \quad \lim_{\Delta x \to 0} Q_r = \lim_{\Delta x \to 0} Q_e = Q,$$

have been introduced.

If only the free vibrations of the beam are of interest, the forcing function p(x,t) is zero, and $\bar{P}$ and $\bar{M}$ become the d'Alembert inertia forces acting per unit length of beam axis.

Due to the postulated linearity of the problem it is permissible to consider bending and shearing distortions separately. The total deflection of the beam is designated by y(x,t), the deflection due to pure bending by $\alpha(x,t)$, the deflection due to pure shear by $\beta(x,t)$.

Thus $$y = \alpha + \beta \qquad (1)$$

The following two differential equations express α in terms of the bending moment M and bending stiffness EI, and β in terms of the shear force Q and the shear stiffness GA_r. E is Young's modulus, G the shear modulus, $G = E/2(1+\nu)$, ν = Poissons's ratio for lateral contraction, $I = I_z$ the moment of inertia of the cross section about the z_1 axis, $A_r = A/k$ the reduced cross section. The value of the reduction factor k will be derived

later.

$$-EI\alpha'' = M \;;$$
$$GA_r\beta' = Q \quad . \tag{2}$$

The inertia force due to the transverse motion of the beam is equal to the negative product of mass density per unit length of beam axis, $\mu(x)$, multiplied by the vertical acceleration $\ddot{\eta}$: $\bar{P} = -\mu\ddot{\eta}$. Before writing down the rotary acceleration term, which is equal to the negative product of the mass moment of inertia per unit length of beam axis (about the z_1-axis) times the angular acceleration, it should be noted that the <u>bending distortion only</u> rotates the beam slices. The shear deformation β merely causes adjacent slices to slide with respect to each other. (This is the simplified concept of the "simple" beam theory!) For small deflections, the angle of rotation is equal to the angle of the tangent to the beam axis, which is in turn approximately equal to the slope α' . Consequently the rotary inertia forces per unit length of beam axis are given by $\bar{M} = -\rho I\ddot{\alpha}'$. [ρ = Massdensity]

With these, the two equilibrium equations of the slice become:

$$Q - M' - \rho I\ddot{\alpha}' = 0 \tag{a}$$

$$Q' \qquad -\mu\ddot{\eta} = 0 \quad . \tag{b}$$

With $\ddot{\alpha}' = \ddot{\eta}' - \ddot{\beta}' = \ddot{\eta}' - \ddot{Q}/GA_r$ eliminate α from (a) :

$$Q - M' - \rho I\ddot{\eta}' + \rho I\ddot{Q}/GA_r = 0 \quad . \tag{c}$$

Differentiate (c) with respect to x:

$$Q' - M'' - \rho(I\ddot{\eta}')' + (\rho/G)\left[(I/A_r)'\ddot{Q} + (I/A_r)\ddot{Q}'\right] = 0, \tag{d}$$

and substitute (b) into (d):

$$\mu\ddot{y} - M'' - \rho(I\ddot{y}')' + (\rho/G)\left[(I/A_r)'\ddot{Q} + (\mu I/A_r)\dddot{y}\right] = 0 . \qquad (e)$$

Express M by y as follows:

$$M = -EI\alpha' = -EI(y'' - \beta'')$$

$$= -EIy''' + EI\,(Q/A_r)'/G$$

$$M = -EIy'' + 2(1+\nu)I\left\{(1/A_r)'Q + \mu\ddot{y}/A_r\right\},$$

differentiate twice with respect to x:

$$M'' = -(EIy'')'' + 2(1+\nu)\left\{(\mu I\ddot{y}/A_r)'' + \left[(1/A_r)'IQ\right]''\right\}, \qquad (f)$$

and introduce (f) into (e):

$$\mu\ddot{y} + (EIy'')'' - 2(1+\nu)(\mu I\ddot{y}/A_r)'' - 2(1+\nu)\left[(1/A_r)'IQ\right]'' - \rho(I\ddot{y}')' +$$

$$+ \rho\left[(I/A_r)'\ddot{Q} + \mu I\dddot{y}/A_r\right]/G = 0 . \qquad (g)$$

Rearranging (g), integrating (b) and introducing it into (g), the following integro-differential equation is obtained:

$$(EIy'')'' - \rho(I\ddot{y}')' + \mu\ddot{y} - 2(1+\nu)\left(\mu I\ddot{y}/A_r\right)'' + (\mu I/A_r)\rho\ddot{y}''/G =$$

$$= 2(1+\nu)\left[(1/A_r)'I\int^x \mu\ddot{y}\,dx\right]'' - (\rho/G)(I/A_r)'\int^x \mu\dddot{y}\,dx \qquad (3)$$

From (3) all of the simplified equations listed in the literature can easily be obtained, for instance:

Free transverse oscillations of a beam, including rotary inertia effects, but neglecting shear distortion. The shear resistance is then infinite, i.e. Poisson's ratio $\mu = -1$, $G = \infty$, and equation (3) becomes:

$$\underline{(EI\eta'')'' - \rho(I\ddot{\eta}')' + \mu\ddot{\eta} = 0} \quad . \tag{4}$$

This equation is listed in K. Hohenemser & W. Prager, Dynamik der Stabwerke, Springer, Berlin 1933.

Free oscillation of a prismatic bar, including rotary inertia and shear effects. Setting EI, μ, A_r constant and carrying out the differentiations in (3), yields:

$$\underline{EI\eta^{IV} - (\rho I + 2(1+\nu)\mu I/A_r)\ddot{\eta}'' + \mu\ddot{\eta} + \rho\mu I\ddot{\eta}''''/GA_r = 0} \; . \tag{5}$$

This equation is listed in St. Timoshenko, Vibration Problems in Engineering, van Nostrand, New York, 1948; where it is integrated for some simple types of boundary conditions.

Assuming harmonic motion:

$$\eta[x,t] = y[x]\, e^{i\sqrt{\lambda}\, t} \tag{6}$$

and introducing (6) into the equation (3) yields the ordinary differential equation for the eigenvalue problem. $\sqrt{\lambda}$ is the characteristic circular frequency of the vibration. The equations (3), (4), and (5) become respectively:

$$\boxed{\begin{aligned}&(EIy'')'' + \lambda\rho(Iy')' - \lambda\mu y + 2(1+\nu)\lambda(\mu Iy/A_r)'' + \lambda^2\rho\mu Iy/GA_r = \\ &= -2(1+\nu)\lambda\left\{I(1/A_r)'\int^x \mu y\, dx\right\}'' - \lambda^2(\rho/G)(I/A_r)'\int^x \mu y\, dx\end{aligned}} \quad . \tag{3'}$$

This is the differential equation of the natural modes, including rotary inertia and shear terms.

$$(EIy'')'' + \lambda[\rho(Iy')' - \mu y] = 0. \qquad (4')$$

This is the differential equation of the natural modes, neglecting shear effects, but with rotary inertia effects included.

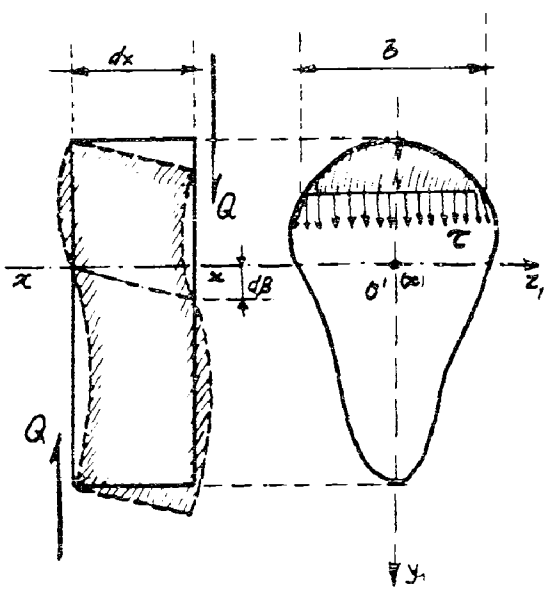
$$EIy'''' + \lambda[(\rho I + 2(1+\nu)\mu I/A_r)y'' - \mu y] + \lambda^2 \rho \mu I y/A_r G = 0. \qquad (5')$$

This is the differential equation of the natural modes for a prismatic bar, including shear and rotary inertia terms.

Equations (4') and (5') have been solved in Hohenemser-Prager's and Timoshenko's texts for the case of uniform beams with rectangular cross section. They show that rotary inertia effects become great at higher frequencies, because then the nodal points are close together and the contribution of the angular accelerations is of the same order of magnitude as that of the transverse accelerations. It seems possible to solve equation (4') also for non-uniform beams, say, for an exponential variation of the height of the beam. The method of variation of parameters will give an analytical solution for the equation under the assumption that λ is known. The real difficulty arisses when the boundary conditions have to be satisfied in order to obtain the characteristic equation for the eigenvalues. An analytical solution of equation (3'), followed by the boundary value problem, appears to be much too difficult and tedious to be attempted here. Furthermore, beams encountered in engineering practice have elastic properties and mass distributions that can not readily be expressed by exponential series and power series and it is concluded that the differential equation approach is

not the appropriate one for the problem. The integral equation methods and variational methods will yield practical solutions. Before these are treated the derivation of the formula of the reduction factor "k" in $A_r = A/k$ is given.

The picture below shows the side - and front view of a slice of the beam acted upon by shearing forces Q only. Let it be assumed that all longitudinal fibres undergo the same angular deformation, i.e. the cross section does not change its shape. Denote by $d\beta$ the shear deflection of an element of length dx. One obtains an equation for $d\beta$ from the fact that the exterior work done by Q during deformation is equal to the strain-energy (interior work) of the shearing stresses produced by Q. Q undergoes a displacement $d\beta$ and increases slowly from zero to its final value Q. Thus the potential energy of Q is equal to:



$$A_e = Q\,d\beta/2 \qquad (a)$$

The strain energy of the shearing forces is given by:

$$A_s = (1/2G)\int_V \tau^2 dV \qquad (b)$$

The volume integral $\int_V \tau^2 dV$ is equal to $dx\int_A \tau^2 dA$ since dV is infinitely thin and τ does not change over the distance dx . The shearing stress intensity is given by:

$$\tau = Q S_z / I_z b ,$$

where S_z is the static moment of the shaded area about the z_1-axis.

Thus, one obtains:

$$A_s = (dx/2G)\int_A \tau^2 dA = (dx/2G)\int_A (Q S_z/I_z b)^2 dA \quad , [dA = dy\,dz]$$

and since Q and I_z are constant for a given x (integration over y and z):

$$A_s = (dx/2G)(Q/I_z)^2 \int_A (S_z/b)^2 dA \quad . \qquad (c)$$

Equating (a) and (c): $A_e = A_s$, there follows:

$$Q d\beta/2 = (dx/2G)(Q/I_z)^2 \int_A (S_z/b)^2 dA$$

and solving for $d\beta$:

$$d\beta = (Q dx/G I_z^2)\int_A (S_z/b)^2 dA \, . \qquad (d)$$

One now sets:

$$\underline{K \equiv (A/I_z)\int_A (S_z/b)^2 dA \, .} \qquad (e)$$

The factor k depends on the cross section only. With it the differential shear deflection is given as:

$$d\beta = K Q dx/GA = Q dx/G(A/K) = Q dx/GA_r .$$

which verifies the equation (2) on page 22.

<u>Values of k for some frequent cross sections:</u>

Rectangle: $k = 6/5$; solid circle: $k = 10/9$; thin-walled circle: $k = 2.0$. For structural I beams one may take with sufficient accuracy the factor $k = 1.0$ and the cross section of the web only: $A/k = A_{web}$. If one takes the total area, k varies from 2.0 to 2.4 for very small and very large sections.

2. Variational Methods. Integral Approach

a) General Comment

In the first section of this chapter the eigenvalue problem has been formulated by means of a differential equation and a set of boundary conditions. This differential equation expresses an equilibrium equation that must hold for any infinitesimal slice of the beam. But this is not the only way of deriving the differential equations. One could have utilized, more elegantly, Hamilton's principle. For the development of the energy methods this principle of least action (as it is called) is the "natural" approach to the eigenvalue problem. A very short and neat development will be given which is taken from G. Hamel, Theoretische Mechanik, Springer, Berlin 1949, and which does not presuppose a knowledge of Langrange's equations. The derivation of the eigenvalue problem follows an article by H. J. Mähly, Die genäherte Berechnung von Eigenwerten elastischer Schwingungen anisotroper Körper, that appeared in vol. 24 of Ergebnisse der exakten Naturwissenschaften, Springer, Berlin 1951, wherein that author investigated the characteristic frequencies of crystals.

The Principle of Hamilton (of least action):

Let dm be an infinitesimal mass element of a system, $\vec{w}$ its vectorial acceleration, $d\vec{K}_e$ an exterior (impressed) force acting on it, and $\delta\vec{r}$ a virtual displacement of the particle. $\vec{r}$ is the radius vector describing the position of the particle, $\vec{v}$ is its velocity which is equal to the rate of change of $\vec{r}$: $\vec{v} = d\vec{r}/dt$.

Then, Lagrange's principle states that the virtual work of the

inertia forces is equal to the virtual work, δA_e, of the impressed forces or

$$\int dm\, \vec{w}\, \delta\vec{r} = \int d\vec{K}_e \cdot \delta\vec{r} = \delta A_e \qquad (a)$$

From it one obtains Lagrange's central equation (as named by Heun) by writing $dm\, w\, \delta\vec{r}$ in a different way, i.e.:

$$\int dm\, \vec{w}\, \delta\vec{r} = \int dm\, \frac{d\vec{v}}{dt}\, \delta\vec{r} = \frac{d}{dt}\int dm\, \vec{v}\,\delta\vec{r} - \int dm\, \vec{v}\, \frac{d\delta\vec{r}}{dt} .$$

But if the virtual displacements are possible $d\,\delta\vec{r} = \delta d\vec{r}$, and

$$\int dm\, \vec{v}\, \frac{d\delta\vec{r}}{dt} = \int dm\, \vec{v}\, \delta\frac{d\vec{r}}{dt} = \int dm\, \vec{v}\,\delta\vec{v} = \frac{\delta}{2}\int dm\, v^2$$

which is equal to the variation of the kinetic energy: δT. Thus there follows the <u>central equation</u>:

$$\int dm\, \vec{w}\, \delta\vec{r} = \frac{d}{dt}\int dm\, \vec{v}\,\delta\vec{r} - \delta T . \qquad (b)$$

Introducing the central equation (b) into Lagrange's principle (a), one obtains:

$$\frac{d}{dt}\int dm\, \vec{v}\,\delta\vec{r} - \delta T = \delta A_e$$

An integration between t_1 and t_2 yields <u>Hamilton's principle</u> in its <u>general form</u>:

$$\int dm\, \vec{v}\,\delta\vec{r}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (\delta T + \delta A_e)\, dt , \qquad (1)$$

If the impressed forces possess a potential, i.e. if δA_e is a total differential $\delta A_e = -\delta U$, then Hamilton's principle appears in a convenient form. Since the δ- process is independent of the time variation, δ may be taken outside the integral sign:

$$\int dm\, \vec{v}\,\delta\vec{r}\Big|_{t_1}^{t_2} = \delta\int_{t_1}^{t_2} (T-U)\, dt = \delta\int_{t_1}^{t_2} L\, dt . \qquad (2)$$

L is the Lagrangian function: $L = T - U$. The integral over L is called the action, whence the name of the principle. The variation $\delta \vec{r}$ can be prescribed to be zero at t_1 and t_2. Then one obtains the usual form of Hamilton's principle:

$$\delta W = \delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \quad , \tag{3}$$

and one says that the integral $W = \int_{t_1}^{t_2} L\, dt$ is stationary in the actual motion of the system.

If the system is an elastic continuum which vibrates freely (uninfluenced by exterior forces) then the potential U is equal to the strain energy of the body. Thus one must express the kinetic energy T and the strain energy U in terms of the deformations (displacements) of the elastic system, introduce these into Hamilton's principle, and thereby obtain an expression which determines the displacements.

b) Formulation of the Eigenvalue Problem

The free vibrations of a body are defined as the simple periodic oscillations which are not influenced by exterior forces. Thus, the deflection of the centroidal axis, which is representative of the deflections of all the particles of the bar (as has been postulated), can be expressed as:

$$y(x,t) = y(x) \sin(\omega t + \psi) = \left\{ \alpha(x) + \beta(x) \right\} \sin(\omega t + \psi), \tag{4}$$

where $y(x)$ was again divided into two parts, $\alpha(x)$ and $\beta(x)$, which represent the bending deflection and shear deflection respectively, viz.:

$$\alpha'' = -\frac{M}{EI} \quad ; \quad \beta' = \frac{Q}{GA_r} \quad . \tag{5}$$

The kinetic energy of the beam is obtained in the following way. Consider a differential slice of the beam. Its length is dx. The mass-density per unit length of beam axis is $\mu = \mu(x)$, the mass moment of inertia of the beam about the z_1-axis (see figure on page 18) per unit length of beam axis is given by $\tau = \tau(x)$. The translatory kinetic energy of the slice is equal to $\mu \dot{y}^2 dx/2$; and the rotary kinetic energy is equal to $\tau \dot{\beta}'^2 dx/2$. (τ is given by ρI_{z_1}.). Thus the total kinetic energy of the beam is expressible as:

$$T = \frac{1}{2} \int (\mu \dot{y}^2 + \tau \dot{\beta}'^2)\, dx \tag{6}$$

The strain energy, in terms of bending moment M and shear force Q, is given as $U = \frac{1}{2}\int M^2 dx/EI + \frac{1}{2}\int Q^2 dx/G A_r$. With equation (5) this can be rewritten in terms of the deflections as follows:

$$U = \frac{1}{2} \int \left\{ EI (\alpha'')^2 + G A_r (\beta')^2 \right\} dx \tag{7}$$

The expressions for kinetic - and strain-energy can be put into a form that shows better that they are quadratic forms in the displacements, viz.:

$$T = \frac{1}{2} V\left[\frac{\partial y}{\partial t}, \frac{\partial y}{\partial t}\right] + \frac{1}{2} R\left[\frac{\partial \beta'}{\partial t}, \frac{\partial \beta'}{\partial t}\right] \tag{8}$$

$$U = \frac{1}{2} B\left[\alpha'', \alpha''\right] + \frac{1}{2} S\left[\beta', \beta'\right] , \tag{9}$$

where the letters V and R are used to designate the kinetic energy due to vertical and rotary motion, and the letters B and S designate the strain-energy due to bending and shear. Furthermore the notation is similar to that used for the inner product of two functions (see page 12). Comparing (8) and (9) with (6) and (7) one finds the definitions:

$$T = \frac{\lambda}{2}\left\{V[y,y] + R[\beta',\beta']\right\} \cdot \cos^2(\omega t + \psi) \tag{11}$$

$$U = \frac{1}{2}\left\{B[\alpha'',\alpha''] + S[\beta',\beta']\right\} \cdot \sin^2(\omega t + \psi) \quad ; \tag{12}$$

where $$\lambda = \omega^2 \quad . \tag{13}$$

Hamilton's principle states that the variation of W must vanish for any time interval $t_2 - t_1$. If one chooses this interval to be an integer multiple of oscillation periods, then the integrals over $\cos^2(\omega t + \psi)$ and $\sin^2(\omega t + \psi)$ are equal, and one is left with the variational equation:

$$\delta\left\{\left(B[\alpha'',\alpha''] + S[\beta',\beta']\right) - \lambda\left(V[y,y] + R[\beta',\beta']\right)\right\} = 0 \quad . \tag{14}$$

This problem is equivalent to that of finding the extrema of the so-called Rayleigh coefficient K(y) which is defined as:

$$K(y) \equiv \frac{B+S}{V+R} \quad , \tag{15}$$

and these are, by virtue of (13), identical with the characteristic values $\lambda_i = \omega_i^2$. For, if one sets out to find the extrema of a quotient (u/v) and designates any such extremum by $\ell = (u/v)_{extr.}$, one knows that the variation of (u/v) must be zero at every extremum, i.e.

$$\delta(u/v) = 0;$$

and carrying out the variation, one obtains:

$$\delta(u/v) = (v\delta u - u\delta v)/v^2$$

Since $v \neq 0$,

$$\delta(u/v) = \left(\delta u - \left(\frac{u}{v}\right)_{extr.}\delta v\right)/v = (\delta u - \ell\,\delta v)/v = 0; \quad \text{or} \quad \delta(u - \ell v) = 0.$$

Thus $\delta(u - \ell v) = 0$ is equivalent to finding the extrema of K(y).

There exist two methods by which the quotient K(y) can be minimized:

(i) One makes K depend on a series of arbitrary parameters a_1, a_2, ...a_n, and determines these parameters in such a way that the function $K(y; a_1,\ldots,a_n)$ is a minimum. This leads to the requirement that n equations:

$$\frac{\partial K}{\partial a_i} = 0 \qquad (16)$$

be satisfied. The equations (16) are a necessary but not sufficient requirement for the existence of a minimum, and it can be demonstrated that they give an upper bound for the higher eigenvalues. (See for instance L. Collatz, Eigenwertproblem, 2. edition, pg. 274 etc.) This method is usually associated with the name of RITZ.

(ii) One tries to make the coefficients of the higher eigen-frequencies $(c_2, c_3, \ldots)$ small with respect to that of the fundamental frequency c_1. This leads to the method of successive approximations (Iteration). After having found the first eigenfrequency one will eliminate its component from the Rayleigh coefficient and then again try to make $c_3, c_4, \ldots$ small in comparison to c_2. Thus one obtains in succession the first, second,.... modes. The great disadvantage of this procedure is the fact that it is very cumbersome to "sweep" (or to free) the Rayleigh coefficient of the components of the modes that have already been iterated. As the numerical labor increases rapidly for the higher modes successive iteration methods will not be treated here.

c) The Ritz Procedure

Let $y(x) = \alpha(x) + \beta(x)$ be expressed in the form of a series

$$y(x;a_1,a_2,\ldots,a_n) = \sum_i^n a_i\, y_i(x) = a_i\, y_i(x) = a_i[\alpha_i(x) + \beta_i(x)] \qquad (17)$$

with $y_i(x) = \alpha_i(x) + \beta_i(x)$.

The functions $y_i(x)$ have to be linearly independent and they must satisfy the conditions of constraint (boundary conditions). Then B,S,V,R become positive definite quadratic forms in the coefficients a_i, (see also Collatz, Eigenwertprobleme) and are given by:

$$\left.\begin{aligned} V[y,y] = V[a,a] &= v_{ij}\, a_i a_j \quad ; \quad \text{with} \quad v_{ij} = V[y_i, y_j] \\ R[\beta',\beta'] = R[a,a] &= r_{ij}\, a_i a_j \quad ; \quad \text{with} \quad r_{ij} = R[\beta_i', \beta_j'] \\ B[\alpha'',\alpha''] = B[a,a] &= b_{ij}\, a_i a_j \quad ; \quad \text{with} \quad b_{ij} = B[\alpha_i'', \alpha_j''] \\ S[\beta',\beta'] = S[a,a] &= s_{ij}\, a_i a_j \quad ; \quad \text{with} \quad s_{ij} = S[\beta_i', \beta_j'] \ . \end{aligned}\right\} \qquad (18)$$

From (10) it is obvious what $v_{ij} \equiv V[y_i, y_j]$ means: it is the value of the following definite integral (to be extended over the beam),

$$v_{ij} = V[y_i, y_j] = \int \mu\, y_i\, y_j\, dx = v_{ji} \quad ; \text{ etc.} \qquad (18')$$

The matrices $(v_{ij}),(r_{ij}),(b_{ij}),(s_{ij})$ are all symmetric. Thus the characteristic values are all real. The Rayleigh coefficient K(, is an extremum if the equations: $\partial K/\partial a_i = 0$ are satisfied, or multiplying both sides with (V + R), one obtains the more convenient form:

$$(V+R)\frac{\partial K}{\partial a_i} = \frac{\partial}{\partial a_i}\left\{(B+S) - \ell(V+R)\right\} = 2\sum_{j=1}^{n}\left[(b_{ij}+s_{ij}) - \ell(v_{ij}+r_{ij})\right]a_j = 0 \quad . \qquad (19)$$

For, if one calls $B + S = D$ and $V + R = H$, one has $K = D/H$ which is the function whose extrema have to be determined. and it is a function of the a_i's.

The system of linear homogeneous equations (19) can have a non-trivial solution only if the determinant of the coefficient matrix is zero. This gives the so-called <u>secular equation</u> (frequency equation):

$$\left.\begin{aligned} &\det\left[u_{ij} - \ell\, t_{ij}\right] = 0 \quad , \\ &\text{where} \quad u_{ij} = (b+s)_{ij} \quad \text{and} \quad t_{ij} = (v+r)_{ij} \end{aligned}\right\} \qquad (20)$$

It should be observed that the coefficients of RITZ's determinant (20) can be obtained <u>directly</u> from the coordinate functions $y_i(x)$, as is evident from equation (18), and it is not necessary to carry out the differentiations with respect to the a_i's. As the matrices (u_{ij}) and (t_{ij}) are symmetric and (t_{ij}) is positive definite, all roots ℓ_i are real and they can be ordered according to size:

$$\lambda_1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \quad .$$

By a known theorem of algebra, the ratios between the coefficients a_{ri} that belong to a Ritz value ℓ_r are equal to the ratios of the cofactors of any row or column of $\| u_{ij} - t_{ij} \|$.

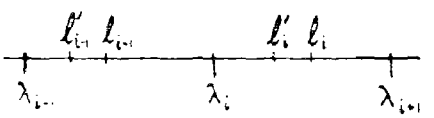
The corresponding solution functions $Y_r(x)$ (or approximations to the natural mode shapes, as they are usually called) are given by:

$$Y_r(x) = a_{ri}\, y_i(x) \tag{21}$$

The determination of the a_{ri} is equivalent to the simultaneous transformation of the matrices (u_{ij}) and (t_{ij}) into diagonal form.

The following lemma is added without proof. (It can be verified in a way similar to that used by H. J. Mähly in vol. 24 of Ergebnisse der exakten Naturwissenschaften).

The Ritz values ℓ_i are greater than the corresponding exact eigenvalues λ_i, but they are not greater than the next higher eigenvalue λ_{i+1}. If one adds another function y_{n+1} to the Ritz expansion, which is linearly independent of all $y_1,\ldots,y_n$, then the Ritz values ℓ'_i of this augmented system can only be smaller than the corresponding ones of the initial system ℓ_i, but they cannot be less than the next lower value λ_{i-1} of the initial system.



It is also possible to prove here that the characteristic vibrations are no longer orthogonal, i.e. that the inner product of any two different ones, multiplied by the square root of the mass density distribution function, does not vanish any more. This proof will be given in the next section, however, where it will be less abstract. The reader can find criteria for the convergence of the procedure in the paper by H. J. Mähly which has already been quoted.

d) The Choice of Coordinate Functions for the Ritz Procedure

It has been shown that Ritz's procedure requires that a set of n linearly independent functions $y_i(x)$ be chosen which must satisfy the conditions of constraint of the problem. The choice of these functions will affect the convergence and the exactitude of the procedure very much. If one had, by pure chance, selected the eigenfunctions as coordinate functions, then the method would be exact. The closer the agreement of the assumed functions y_i is with the eigenfunctions, the better the convergence will be. It is now very easy to find sets of linearly independent functions - that are even orthogonal - which satisfy all boundary conditions and which are very close to the actual eigenfunctions, i.e. the eigenfunctions of a uniform beam, neglecting shear and rotary inertia effects, with the same conditions of constraint as the one to be analyzed. (K. Hohenemser & W. Prager, Dynamik der Stabwerke, Springer 1933, have discussed this at great length and have shown, by many examples of structures with very uneven distribution of mass density and moment of inertia per unit length, that the so-called uniform beam modes are the best coordinate functions to use.)

Tables for uniform beam modes can be found in many text books, notably Lord Rayleigh, Theory of Sound, and in the text mentioned above. Extensive tables are added to this report.

Equation (4) shows that it is also necessary to know the bending deflection and shearing deflection separately, and thus the problem arises as to how to obtain $\alpha(x)$ and $\beta(x)$, if their sum $y(x)$ together with the elastic properties of the beam are given. From equation (5) one notices that

both M and Q have to be obtained, in order that α and β can be obtained by integrations. In terms of a loading intensity p(x) per unit length of beam axis, Q and M are given as first and second integrals of p:

$$Q = -\int^x p\,dx \quad ; \quad M = \int^x Q\,dx = -\int^x dx \int^x p\,dx \quad . \tag{22}$$

Differentiating (4) twice with respect to x, together with (5) and (22), one obtains the following general relation between deflection y(x) and load intensity p(x):

$$y(x) = \int^x dx \int^x (dx/EI) \int^x dx \int^x p\,dx - \int^x (dx/GA_r) \int^x p\,dx \tag{23}$$

which reduces, for prismatic bars, to

$$\frac{d^2y}{dx^2} = -\frac{1}{EI}\left\{M + 2(1+\nu)\frac{I}{A_r}p\right\} = -\frac{1}{EI}\left\{-\int^x dx \int^x p\,dx + \frac{EI}{GA_r}p\right\} \quad . \tag{23'}$$

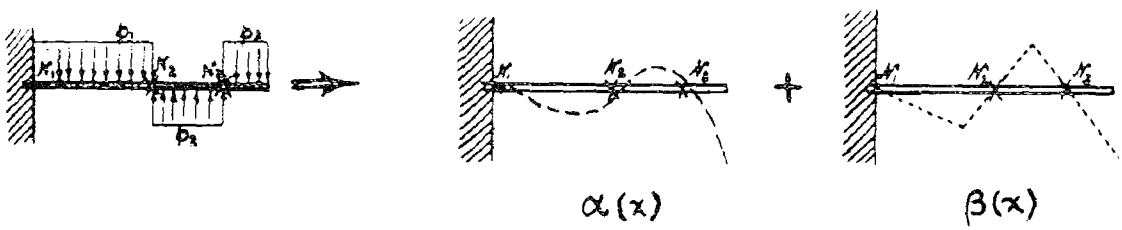
(See for instance S. Timoshenko, Strength of Materials, vol. 1, pg. 171)

The solution of (23) for p(x) with a given y(x) would be a very cumbersome undertaking and could, in general, not be carried out analytically. But remembering that the Ritz procedure requires only functions that satisfy the boundary conditions, and that the uniform beam modes are utilized because they insure rapid convergence, one can modify the problem slightly and obtain $\alpha(x)$ and $\beta(x)$ by the two following procedures:

(1) Take the mode-loads of the uniform beam. These are the static loads that produce the mode deflections. From these one obtains the bending deflection α and shear deflection β. (The first eight mode-loads for uniform beams with various boundary conditions are tabulated in this report.)

(ii) If uniform beam mode-loads are not available, estimate the location of the nodal points and apply alternate uniform loads between these nodes. Then, calculate the bending - and shearing deflections under these loads and use them in the Ritz expansion.

The following picture illustrates how the third of the linearly independent functions, $y_3(x) = \alpha_3(x) + \beta_3(x)$ can be obtained.



(The load-intensities p_1, p_2, p_3 can be equal, or can be determined such that the points N_2 and N_3 do not undergo any static deflection.)

The underlying idea of this procedure is that these deflection lines will roughly represent the natural modes and thus one is assured that the RITZ values ℓ_i are close to the actual λ_i and that the procedure converges rapidly. The expansion coefficients (or participation factors) of the r-th linearly independent function $y_r(x)$ in the r-th natural mode:

$$Y_r(x) = a_{ri}\ y_i\ (x)$$

will then be such that the factor a_{rr} is much larger than the other a_{rj}'s (where $j \neq r$). This means that the matrix (a_{ij}) is almost a diagonal matrix. It is, however, not necessary to choose the n linearly independent functions in this way, and one could, for instance, also select the deflection lines

due to a point load that is stationed successively in n different points. (It is interesting to know that the n-th natural frequency of vibrating beams is completely characterized by the location of the nodal points. In beams with both ends supported one must not count one of these. The proof is given by K. Hohenemser & W. Prager in ZAMM, 1931, vol. 11, #2)

The RITZ procedure offers the advantage that it provides the analyst with relatively good approximations to the frequencies and the mode shapes. It has the disadvantage that the error in the approximation cannot be estimated, and also that, if one adds another function y_{n+1} to the set of linearly independent function $\{y_n\}$, all the labor of solving the frequency equation (20) has to be repeated. It has been mentioned that this addition of another coordinate function can only improve the results.

3. Integral Equation Methods

a) Influence Coefficients

Consider a beam as defined in section 1 of this chapter. If this beam is acted upon by a concentrated load P which is perpendicular to the beam axis and located at a point designated by "j", then it produces at a point "i" a deflection δ_{ij} and an angular rotation θ_{ij}. Similarly, a couple at "j" causes a deflection ϑ_{ij} and an angular rotation ψ_{ij} at "i". The concentrated load and the couple shall both have the magnitude one.

Formulas for these influence coefficients can easily be derived from the equations of virtual work. Let M_k, Q_k (and $\bar{M}_k$, $\bar{Q}_k$) be the bending moment and shear force in the beam due to a unit force (and a unit couple) at "k". Then the above listed influence coefficients are given by the following integrals to be extended over the beam:

$$\left.\begin{aligned}
\delta_{ij} &= \int (M_i M_j/EI)\,dx + \int (Q_i Q_j/GA_r)\,dx \quad , \\
\theta_{ij} &= \int (\bar{M}_i M_j/EI)\,dx + \int (\bar{Q}_i Q_j/GA_r)\,dx \quad , \\
\vartheta_{ij} &= \int (M_i \bar{M}_j/EI)\,dx + \int (Q_i \bar{Q}_j/GA_r)\,dx \quad , \\
\psi_{ij} &= \int (\bar{M}_i \bar{M}_j/EI)\,dx + \int (\bar{Q}_i \bar{Q}_j/GA_r)\,dx \quad .
\end{aligned}\right\} \qquad (1)$$

They obey Maxwell's reciprocal laws:

$$\delta_{ij} = \delta_{ji}; \quad \psi_{ij} = \psi_{ji}; \quad \theta_{ij} = \vartheta_{ji} ,$$

as is evident from (1). If the point "i" has the coordinate $x_i = x$ and "j" the coordinate $x_j = \xi$, and if both x and ξ are considered to be independent variables, then the four influence coefficients become influence functions.

b) Influence Functions (Green's Functions)

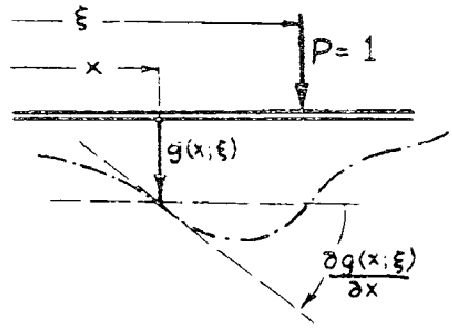
Let the influence function corresponding to the coefficient δ_{ij} be designated by g. It is clearly a function of x and ξ : $g = g(x; \xi)$, and in view of Maxwell's law it is symmetric in the variables, hence:

$$g(x; \xi) = g(\xi; x) .$$

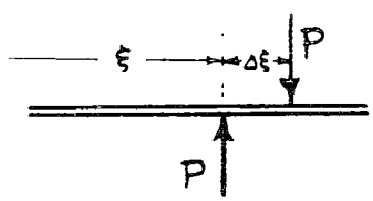
The determination of $g(x; \xi)$ is sufficient, for, the remaining influence functions can be obtained by partial differentiations.

Lemma 1: The influence function of angular rotation due to a point load is equal to the partial derivative of $g(x; \xi)$ with respect to x.

The following sketch illustrates this.



In order to obtain the deflection influence due to a unit moment, observe that a unit moment can be considered as the limiting case of a couple, $P\Delta\xi = \mathfrak{M}$, which is subjected to the restriction: $\lim_{\Delta\xi\to 0}(P\Delta\xi) = \mathfrak{M}$.



The deflection at x due to a load P at $\xi + \Delta\xi$ is given by $g(x;\ \xi + \Delta\xi\ )\ P$, and thus the deflection at x due to the couple $P\ \Delta\xi$ is equal to $P\{g(x;\xi+\Delta\xi) - g(x;\xi)\}$. With $P = \mathfrak{M}/\Delta\xi$ this takes the form: $\mathfrak{M}\{g(x;\xi+\Delta\xi) - g(x;\xi)\}/\Delta\xi$. In the limit $\Delta\xi \to 0$ the deflection at x due to a moment $\mathfrak{M}$ at ξ :

$$\lim_{\Delta\xi\to 0} \mathfrak{M}\ \frac{g(x;\xi+\Delta\xi) - g(x;\xi)}{\Delta\xi} = \mathfrak{M}\ \frac{\partial g(x;\xi)}{\partial\xi}\ ;$$

consequently:

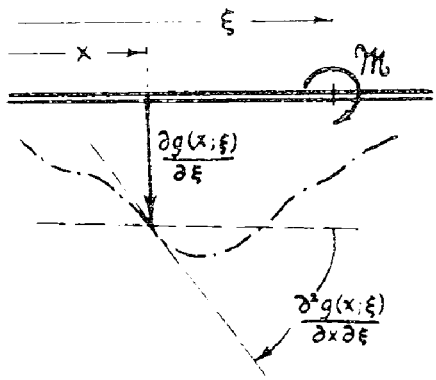
Lemma 2: The deflection influence function due to a unit moment at ξ is equal to the partial derivative with respect to ξ of the deflection influence function for a unit force at ξ .

Finally:

Lemma 3: The influence function for the angular deflection due to a unit moment, is equal to the partial derivative with respect to x and

ξ of the deflection influence function for a unit force at ξ .

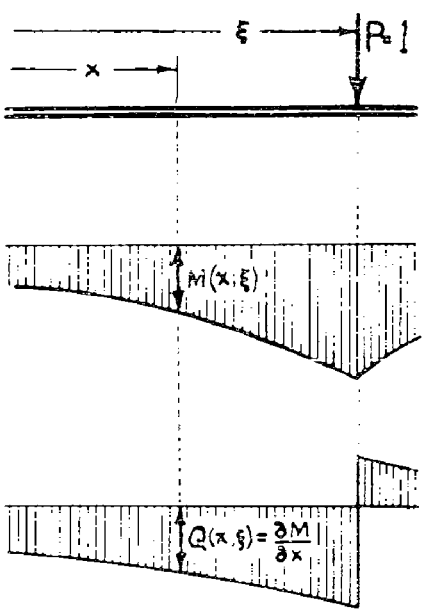
The following sketch illustrates this.



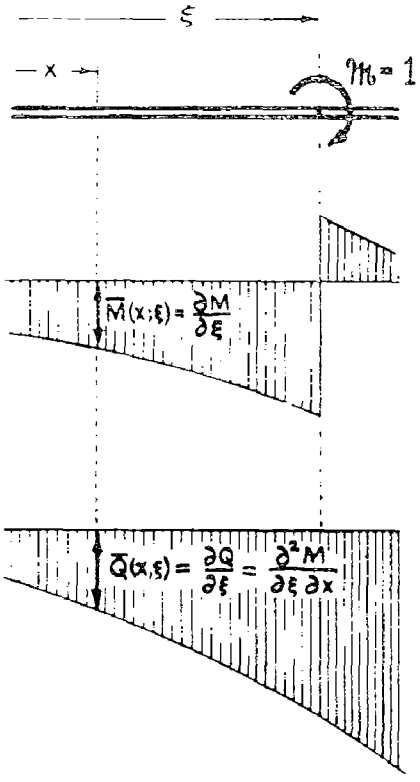
It is pointed out that the relations just derived are not limited to the Green's functions for the deformations. They are equally valid for those of the shear force and bending moment.

Lemma 4: The influence functions for the shear force and the bending moment of a beam which is under the action of a unit couple located at $x = \xi$ are the partial derivatives with respect to ξ of the corresponding influence functions due to a unit force at ξ .

The proof follows from the remark that one has only to require that the limiting value of the couple $P\Delta\xi$ be finite and equal to $\mathfrak{M}$.



The adjacent figures show first a beam with a unit force at ξ and the corresponding bending moment - and shear-diagrams, in which a value at x is marked $M(x;\xi)$ and $Q(x;\xi)$.



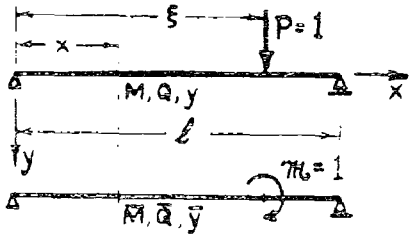
Next is the same beam with a unit couple at ξ and with the corresponding diagrams for the bending moment $\bar{M}$ and the shear force $\bar{Q}$.

Note that the shear force is always the x-derivative of the bending moment, it is thus only necessary to obtain the Green's function of the bending moment due to a unit force. The others are then partial derivatives of it.

Sometimes the evaluation of the Green's function due to a unit couple is easier than the calculation of the Green's function due to a unit force, followed by partial differentiations. Then one will, of course, determine it directly.

Formulas for the Green's functions of uniform beams are compiled in R. Roark, Formulas for Stress and Strain, McGraw Hill, New York 1943. The following example was taken from this book.

Illustration:



uniform beam, simply supported, acted upon by a unit force, and also by a unit couple. The Green's functions, in the interval from zero to ξ, are given by:

$$M(x;\xi) = x(\ell-\xi)/\ell \qquad \bar{M} = -x/\ell$$

$$Q(x;\xi) = (\ell-\xi)/\ell \qquad \bar{Q} = -1/\ell$$

$$y(x;\xi) = x(\ell-\xi)\left[2\ell\xi-\xi^2-x^2\right]/6EI\ell \qquad \bar{y} = -x\left[6\xi\ell-3\xi^2-2\ell^2-x^2\right]/6EI\ell \ .$$

One verifies here readily that the quantities indicated by a bar are the ξ - derivatives of those without a bar.

c) Formulation of the Integral Equation

Suppose that the beam oscillates freely. As shown in section 1, every slice located at $x = \xi$ is acted upon by inertia loadings which are composed of the inertia force $-\mu\ddot{y}d\xi$ due to the transverse acceleration $\ddot{y}$ and the inertia moment $-\tau\ddot{\alpha}'d\xi$ due to the rotary acceleration $\ddot{\alpha}'$. Let the deflection influence function due to a unit load at ξ be $g(x;\xi)$ as has just been established in section b). Then the load $-\mu\ddot{y}d\xi$ at ξ produces at x a differential deflection: $-g(x;\xi)\mu\ddot{y}d\xi$.

The deflection influence function $g(x;\xi)$ is made up of two parts, the deflection due to bending distortion only; $\alpha(x;\ )$ and the deflection due to shearing distortion only: $\beta(x;\xi)$. Thus,

$$g(x;\xi) = \alpha(x;\xi) + \beta(x;\xi) \ .$$

According to Lemma #2 of the previous section, the deflection at x due to a unit moment at ξ is equal to: $\partial g(x;\xi)/\partial\xi$, and consequently, the inertia moment $-\tau\ddot{\alpha}'d\xi$ at ξ produces a differential deflection: $-\frac{\partial g(x;\xi)}{\partial\xi}\tau\ddot{\alpha}'d\xi$ at x. The sum of these two differential effects is equal to:

$$dy = -\left\{ g(x;\xi)\mu\ddot{y}\,d\xi + \frac{\partial g(x;\xi)}{\partial \xi}\tau\ddot{\alpha}'\,d\xi \right\} .$$

One obtains the total deflection y(x) integrating over all the differential effects:

$$y(x;t) = -\int_a^b \left\{ g(x;\xi)\mu(\xi)\ddot{y}(\xi) + \frac{\partial g(x;\xi)}{\partial \xi}\tau(\xi)\ddot{\alpha}'(\xi) \right\} d\xi .$$

The natural modes are the free harmonic vibrations of the beam: $y(x;t) = y(x)\sin(\omega t + \psi)$ and, with $\omega^2 = \lambda$, one arrives at the integral equation for the natural modes of the beam:

$$y(x) = \lambda \int_a^b \left\{ g(x;\xi)\mu(\xi)y(\xi) + \frac{\partial g(x;\xi)}{\partial \xi}\tau(\xi)\frac{d\alpha(\xi)}{d\xi} \right\} d\xi . \qquad (2)$$

Equation (2) is a homogeneous, non-linear integro-differential equation of the Fredholm type. The function $\alpha(x)$ depends, as has been shown, in a complicated way on y(x).

The equation can be simplified considerably if one sets: $\alpha(x) \approx y(x)$, i.e. if one neglects the shear effects in the rotary inertia terms. This is justifiable, since for small vibrations and lower modes the functions y(x) and $\alpha(x)$ differ only very little and their respective first derivatives are practically alike. Furthermore the approximation occurs in a term which is only a correction term and much smaller than the first expression in the integrand which represents the deflection due to transverse motion of the beam only. (This question will be discussed in more detail in chapter IV.) With

the simplication $d\alpha/dx = dy/dx$, equation (2) appears as

$$y(x) = \lambda \int_a^b \left\{ g(x;\xi)\mu(\xi)y(\xi) + \frac{\partial g(x;\xi)}{\partial \xi}\tau(\xi)\frac{dy(\xi)}{d\xi} \right\} d\xi \quad . \tag{3}$$

One can eliminate the derivative $dy/d\xi$ in the integrand by partial integration of the second term on the right and obtains then:

$$y(x) = \lambda \left\{ \int_a^b g(x;\xi)\mu(\xi)y(\xi)\,d\xi - \int_a^b \frac{\partial}{\partial \xi}\left[\tau(\xi)\frac{\partial g(x;\xi)}{\partial \xi}\right] y(\xi)\,d\xi + \left[\frac{\partial g(x;\xi)}{\partial \xi}\tau(\xi)\,y(\xi)\right]_{\xi=a}^{\xi=b} \right\} ,$$

or also:

$$y(x) = \lambda \left\{ \int_a^b \left[g(x;\xi)\mu(\xi) - \frac{\partial}{\partial \xi}\left(\tau(\xi)\frac{\partial g(x;\xi)}{\partial \xi}\right)\right] y(\xi)\,d\xi + \left[\frac{\partial g(x;\xi)}{\partial \xi}\tau(\xi)y(\xi)\right]_{\xi=a}^{\xi=b} \right\} . \tag{3'}$$

Introduce the abbreviations:

$$G(x;\xi) = g(x;\xi)\mu(\xi) - \frac{\partial}{\partial \xi}\left[\tau(\xi)\frac{\partial g(x;\xi)}{\partial \xi}\right] = g(x;\xi)\mu(\xi) - \frac{\partial H(x;\xi)}{\partial \xi} \tag{4}$$

$$H(x;\xi) = \tau(\xi)\frac{\partial g(x;\xi)}{\partial \xi} \quad ,$$

and recall the notation of the inner product of two functions, then (3') can be written in the compact form:

$$y = \lambda \left[(G,y) + H\,y \right]_a^b \tag{5}$$

Note well that G is a function of x and ξ , a Green's function which is <u>not symmetric</u>, and y is a function of ξ also, if it is in a round bracket. This bracket is a short notation for the definite integral of the product of the factors inside, and, since G is also a function of x, which is not affected by the integration process, (G,y) must be a function of x. Similarly H is a function of x and ξ , and y is again a function of ξ , whenever it is multiplied with another function. The vertical bar $\big|_a^b$ indicates that the difference of the values of the function Hy for $\xi = b$, and $\xi = a$ is

to be taken. Since x is not involved in this process of subtracting the lower limit from the upper one, $Hy\Big|$ is indeed a function of x. Consequently, both sides of equation (5) are functions of x alone. If the simplification $\alpha \approx y$ is not made, equation (5) reads:

$$y(x) = \lambda \left\{ \int_a^b g(x;\xi)\mu(\xi)y(\xi)\,d\xi - \int_a^b \frac{\partial}{\partial \xi}\left[\tau(\xi)\frac{\partial q(x;\xi)}{\partial \xi}\right]\alpha(\xi)\,d\xi + \left[\frac{\partial q(x;\xi)}{\partial \xi}\tau(\xi)\alpha(\xi)\right]_{\xi=a}^{\xi=b} \right\} . \quad (5')$$

From equation (5) one proves now without difficulty that the characteristic functions are not orthogonal, for, let y_i and y_j be two eigenfunctions and λ_i and λ_j the corresponding eigenvalues, then the two following equations must be satisfied:

$$y_i/\lambda_i = (G, y_i) + Hy_i\Big| ,$$

$$y_j/\lambda_j = (G, y_j) + Hy_j\Big| .$$

Take now the inner product of the first equation with respect to y_j, also the inner product of the second one with respect to y_i, subtract the second one from the first one, and note that $(y_i, y_j) = (y_j, y_i)$ then one obtains:

$$\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right)(y_i, y_j) = \big((G, y_i), y_j\big) + \big(Hy_i\big|, y_j\big) - \big((G, y_j), y_i\big) - \big(Hy_j\big|, y_i\big) .$$

If one writes this out in detail, it becomes:

$$\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right)\int_a^b y_i y_j\,dx = \int_a^b\int_a^b G(x;\xi)\,y_i(\xi)\,y_j(x)\,d\xi\,dx - \int_a^b\int_a^b G(x;\xi)\,y_j(\xi)\,y_i(x)\,d\xi\,dx$$
$$+ \int_a^b \left[H(x;\xi)\,y_i(\xi)\right]_{\xi=a}^{\xi=b} y_j(x)\,dx - \int_a^b \left[H(x;\xi)\,y_j(\xi)\right]_{\xi=a}^{\xi=b} y_i(x)\,dx ,$$

Inverting the dummy variables in the second term, and combining then these

two integrals by factoring out $y_i(\xi)y_j(x)$, one gets:

$$\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_j}\right)\int_a^b y_i y_j\,dx = \int_a^b\int_a^b \Big[G(x;\xi)-G(\xi;x)\Big] y_i(\xi)\,y_j(x)\,d\xi\,dx \qquad (6)$$

$$+\int_a^b \left(\left[H(x;\xi)\Big\{y_i(\xi)\,y_j(x)-y_j(\xi)\,y_i(x)\Big\}\right]_{\xi=a}^{\xi=b}\right)dx\,.$$

$$(\text{for } i\neq j)$$

If the beam were supported at both ends, then y would be zero at the ends and consequently the second integral of the right hand side of (6) would vanish. Since $G(x;\xi) \neq G(\xi;x)$, one concludes that for this case the orthogonality cannot exist. If the beam is clamped at one end, then the second integral of the right hand side of (6) will not vanish, and, in general, orthogonality cannot exist. In a similar way one verifies from (5') that the eigenfunctions are not orthogonal.

Only if <u>rotary inertia</u> effects are <u>neglected</u>, can one obtain orthogonality with respect to a weighting function $\mu(x)$, for in this case, equation (2) reduces to:

$$y(x) = \lambda\int_a^b g(x;\xi)\mu(\xi)\,y(\xi)\,d\xi \qquad (7)$$

With the new function $\bar{y} = \mu y$ this becomes:

$$\bar{y}/\mu = \lambda\int_a^b g(x;\xi)\,\bar{y}(\xi)\,d\xi$$

and if $\bar{y}_i$, $\bar{y}_j$ and $\bar{\lambda}_i$, $\bar{\lambda}_j$ are two different eigenfunctions and their corresponding eigenvalues of the problem, there follows:

$$\left(\frac{1}{\bar{\lambda}_i}\right)\left(\bar{y}_i/\mu\right) = \int g(x;\xi)\bar{y}_i(\xi)\,d\xi \quad ; \quad \left(\frac{1}{\bar{\lambda}_j}\right)\left(\bar{y}_j/\mu\right) = \int g(x;\xi)\,\bar{y}_j(\xi)\,d\xi$$

Multiplying the first equation by $\bar{y}_j(x)$, the second by $\bar{y}_i(x)$, integrating over x between the limits a and b, and subtracting now the second from the first one of the altered equations, yields:

$$\left(\frac{1}{\bar{\lambda}_i} - \frac{1}{\bar{\lambda}_j}\right)\int_a^b \bar{y}_i\,\bar{y}_j\,\frac{dx}{\mu} = \int_a^b\int_a^b \left\{g(x;\xi)\,\bar{y}_i(\xi)\,\bar{y}_j(x) - g(x;\xi)\,\bar{y}_j(\xi)\,\bar{y}_i(x)\right\} d\xi\,dx \ .$$

Interchanging the dummy variables x and ξ in the second product and combining it with the first one, there follows:

$$\left(\frac{1}{\bar{\lambda}_i} - \frac{1}{\bar{\lambda}_j}\right)\int_a^b \frac{\bar{y}_i\,\bar{y}_j}{\mu}\,dx = \int_a^b\int_a^b \left\{g(x;\xi) - g(\xi;x)\right\} y_i(\xi)\,y_j(x)\,d\xi\,dx \ . \qquad (8)$$

In section b) it has been shown that $g(x;\xi) = g(\xi;x)$; consequently the right-hand side of (8) is zero, and one finds, that for $i \neq j$

$$\int_a^b \frac{\bar{y}_i\,\bar{y}_j}{\mu}\,dx = 0 \quad ;$$

with $\bar{y}_i = \mu\,y_i$ and $\bar{y}_j = \mu\,y_j$ this equation yields the generalized orthogonality relation (orthogonality with respect to the weighting function μ):

$$\int_a^b \mu(x)\,y_i(x)\,y_j(x)\,dx = 0 \qquad (9)$$

or one may also state that the functions $\sqrt{\mu}\,y_i$ and $\sqrt{\mu}\,y_j$ are orthogonal in the conventional sense. With the terminology of chapter II, section 2, it is concluded that the eigenfunctions of the vibrating beam, neglecting rotary inertia effects, multiplied by the square root of the mass density distribution function form an orthogonal set, which can readily be normalized. For, if: $\left(\sqrt{\mu}\,y_i\,,\sqrt{\mu}\,y_i\right) = N_{y_i}$ then:

$$\left(\sqrt{\frac{\mu}{N_{y_i}}}\,y_i\,,\sqrt{\frac{\mu}{N_{y_j}}}\,y_j\right) = e_{ij} \quad , \quad (e_{ii} = 1 \ , \ e_{ij} = 0 \ \text{for} \ i \neq j) \qquad (10)$$

which is the statement that the set $\left\{\sqrt{\frac{\mu}{N y_n}}\, y_n\right\}$ is ortho-normal.

d) Approximate Solutions of the Integral Equation

The integral equation (5), or (5'), can in general not be solved exactly, and the problem arises as to how to find approximate solutions of it. Assume that the exact solution can be approximated by a linear aggregate of linearly independent solutions $y_j(x)$ - or $\alpha_j(x)$ - which satisfy individually the boundary conditions of the problem.

This is again a RITZ procedure. So let:

$$y(x) \approx \bar{y}(x) \equiv a_j y_j(x) = a_j\left[\alpha_j(x) + \beta_j(x)\right] \tag{11}$$

and introduce it into equation (5), which was:

$$y = \lambda\left\{(G,y) + Hy\right\} \quad . \tag{5}$$

This yields: $$a_j y_j \approx \lambda\left\{(G, a_j y_j) + H a_j y_j\right\} = \lambda a_j\left\{(G, y_j) + H y_j\right\} \tag{12}$$

and one requires now that the left hand side coincide with the right hand side at as many points x_j as there are unknown coefficients a_j, i.e. at n points. (See section f) for motivation.) This gives a set of n linear homogeneous equations for the coefficients a_j:

$$a_j y_j(x_i) = \lambda a_j\left\{\left(G[x_i;\xi], y_j[\xi]\right) + H[x_i,\xi]\, y_j[\xi]\Big|_{\xi=a}^{\xi=b}\right\} \quad ;$$

with the abbreviations:

$$\mathcal{Y} = (y_{ij}) = \left(y_j[x_i]\right) \quad , \qquad \mathfrak{a} = \left\{a_1, a_2, \cdots, a_n\right\} \quad ,$$

$$G = (G_{ij}) = \left(\int G[x_i;\xi] \quad y_j[\xi] \; d\xi \right) ,$$

$$H = (H_{ij}) = \left[H(x_i;\xi)\, y_j(\xi) \right]_{\xi=a}^{\xi=b} ,$$

$$L = G + H ,$$

(13)

one obtains the following generalized matrix eigenvalue equation for the vector a:

$$ya = \lambda \, La \tag{14}$$

This equation can be reduced to an ordinary eigenvalue equation by premultiplying both sides with the inverse of the matrix L : L^{-1}, which exists, since $\| L \|$ is not singular; thus with $L^{-1}L = E$, $Ea = a$

$$L^{-1} ya = \lambda \, a$$

and with $L^{-1}y = B$, where B is the solution of the equation:

$$Lx = y \qquad B = X , \tag{15}$$

one obtains the ordinary matrix eigenvalue equation:

$$Ba = \lambda \, a . \tag{16}$$

This homogeneous equation has a solution only if the determinant of the coefficient matrix vanishes:

$$\det\left[\mathbf{y} - \lambda \mathbf{L}\right] = \left\| y - \lambda L \right\| = 0 \quad , \tag{17}$$

Equation (17) is the generalized characteristic equation, or the lambda equation, and, written out in detail, reads:

$$\begin{vmatrix} y_{11} - \lambda l_{11}, & y_{12} - \lambda l_{12}, & \cdots & , & y_{1n} - \lambda l_{1n} \\ \cdots & \cdots & \cdots & & \cdots \\ y_{n1} - \lambda l_{n1}, & y_{n2} - \lambda l_{n2}, & \cdots & , & y_{nn} - \lambda l_{nn} \end{vmatrix} = 0$$

It is an algebraic equation in λ whose coefficient of λ^n is equal to $\|\mathbf{L}\|$ and whose absolute term is $\|\mathbf{y}\|$. Thus it is an equation of the n-th degree in λ, if the matrix $\mathbf{L}$ is not singular. With this condition equation (16) exists and one can obtain the characteristic frequencies from the secular equation:

$$\|\mathbf{B} - \lambda \mathbf{E}\| = 0, \quad \text{or} \quad \begin{vmatrix} b_{11} - \lambda & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} - \lambda & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} - \lambda \end{vmatrix} = 0 \quad , \tag{18}$$

which is the necessary condition that (16) possesses non-trivial solutions. The matrix $\mathbf{B}$ is best obtained by solving the set of linear equations (15).

It is now necessary to examine how the matrix $\mathbf{L}$, eq.(13), is constructed. From eq. (4) one finds:

$$G(x;\xi) = g(x;\xi)\mu(\xi) - \frac{\partial H(x;\xi)}{\partial \xi} \quad ; \qquad H(x;\xi) \equiv \tau(\xi)\frac{\partial g(x;\xi)}{\partial \xi}$$

where $g(x;\xi)$ is the deflection influence function due to a unit load at ξ. Equation (3') states what integration has to be carried out. The ij-th element of $\mathbf{G}$, G_{ij}, is the value of the following integral at a point with the coordinate $x = x_i$ and with the j-th coordinate function $y_j(x)$:

$$G_{ij} = \int_a^b \left\{ g(x_i;\xi)\mu(\xi) - \frac{\partial}{\partial \xi}\left[\tau(\xi)\frac{\partial g(x_i;\xi)}{\partial \xi}\right]\right\} y_j(\xi)\, d\xi \quad . \tag{19}$$

The points x_i are taken at regularly spaced intervals over the beam axis. Let the point $x = a$ correspond to $i = 1$, and $x = b$ to $i = n$; i.e., let the length $(b - a)$ be divided into "n-1" equal intervals. From (19) it is apparent that every G_{ij} is the difference of two values, namely the value of the integral at the upper limit minus the value of the integral at the lower limit. But H_{ij} is a similar difference, viz.

$$H_{ij} = \left[\tau(\xi)\,\frac{\partial g(x_i;\xi)}{\partial \xi}\, y_j(\xi)\right]_{\xi=a}^{\xi=b} \tag{20}$$

and consequently G_{ij} and H_{ij} can be combined to yield ℓ_{ij} which appears, written out in full, as:

$$\mathbf{L} = (\ell_{ij}) ; \qquad \ell_{ij} = \left[\int^{\xi} \left\{ g(x_i;\xi)\mu(\xi) - \frac{\partial}{\partial \xi}\left[\tau(\xi)\frac{\partial g(x_i;\xi)}{\partial \xi}\right]\right\} y_j(\xi)\, d\xi + \tau(\xi)\frac{\partial g(x_i;\xi)}{\partial \xi}\, y_j(\xi)\right]_{\xi=a}^{\xi=b} . \tag{21}$$

The procedure, outlined so far, is known as <u>collocation using assumed functions</u>. It is, of course, not the only possible method for determining the coefficients a_j as well as possible.

<u>Other Methods for Determining the Coefficients a_j</u>:

A more accurate result is obtained - but with more work - if one requires that not the ordinates of the approximating function be equal to the values calculated from equation (12) at n discrete points (usually equally spaced), but that the integrals over the n subintervals (usually of equal length) be equal, i.e. the integral over the k-th subinterval of the left-hand side of equation (12) shall be equal to the corresponding one of the right-hand side of this equation. The integration process has the effect of making the mean values of the two functions of the left-and right-hand side equal in n

subintervals. Formally this yields the following set of n linear homogeneous equations:

$$a_j \int_{x_{i-1}}^{x_i} y_j \, dx = \lambda a_j \int_{x_{i-1}}^{x_i} \left[(G, y_j) + H y_j \right] dx \tag{22}$$

where: $\sum_{i}^{n} x_i - x_{i-1} = b - a$, and usually $x_i - x_{i-1} = (b - a)/n$.

One can, of course, write the values of the integrals over the n subintervals as elements of matrices,

$$C_{ij} \equiv \int_{x_{i-1}}^{x_i} y_j \, dx \quad ; \quad d_{ij} \equiv \int_{x_{i-1}}^{x_i} \left[(G, y_j) + H y_j \right] dx \ , \tag{23}$$

and put (22) into the form: $\mathbf{C a} = \lambda \mathbf{D a}$

or, $\mathbf{F a} = \lambda \mathbf{a}$ with $\mathbf{F} = \mathbf{D}^{-1}\mathbf{C}$. (24)

The frequency equation associated with (22) or (24) is given by:

$$\| \mathbf{C} - \lambda \mathbf{D} \| = 0 \quad \text{or} \quad \| \mathbf{F} - \lambda \mathbf{E} \| = 0 \ . \tag{25}$$

Still another method for determining the factors a_j is one based upon lemma 2 of chapter II. It was shown there that a function is uniquely determined by its expansion coefficients, particularly that the difference of two equal functions (called the zero-function) must have vanishing expansion coefficients. From equation (II-22) it follows that these are zero, if the components of the zero-function with respect to all the members of a complete set of functions vanish.

(For, $\mathfrak{a} = \mathfrak{B}^{-1}\mathfrak{C} = 0$, or $a_\nu = b_{\nu\mu}^{-1} c_\mu$ if $c_\mu = (f, \eta_\mu) = 0$, since $\mathfrak{B}^{-1} \neq 0$.)

In the present case, it is desired to make $a_j y_j$ equal to $\lambda a_j\{(G,y_j) + Hy_j|\}$ or, which is the same, their difference equal to zero:

$$a_j y_j - \lambda a_j\{(G,y_j) + Hy_j|\} = 0.$$

Thus, if $\{\varphi_n(x)\}$ is a set of linearly independent functions,

$$\left(a_j y_j - \lambda a_j\{(G,y_j) + Hy_j|\}, \varphi_i\right) = 0, \quad i = 1,2,\ldots,n,\ldots \tag{26}$$

must be satisfied for every i, and the a_j have to be determined accordingly. This equation can also be written as

$$a_j(y_j,\varphi_i) = \lambda a_j\left((G,y_j) + Hy_j|, \varphi_i\right) \tag{27}$$

where i and j must have the same range in order that the a_j's can be determined. If both $\{y_j\}$ and $\{\varphi_i\}$ were complete sets (with infinitely many members), this procedure would give the exact solution, provided the series $a_j y_j$ were absolutely convergent. If only the first n terms of the sets are taken, one obtains an approximation to the exact value.

Setting:

$$h_{ij} = (y_j,\varphi_i) \quad ; \qquad k_{ij} = \left((G,y_j) + Hy_j|, \varphi_i\right) \quad ; \tag{28}$$

equation (27) can be written as:

$$\mathbf{Ha} = \lambda \mathbf{Ka} \quad . \tag{29}$$

The particular choice of the complete set has a decisive influence upon the quality of the approximation. The process just described is usually called the method of weighting functions, and it is said that in eq. (27) the inner products of both sides with the weighting functions φ_i must be equal.

Examples of sets of weighting functions:

An easy set to work with are the functions,

$$x^0 , x^1, x^2, \ldots , x^{n-1}; \tag{30}$$

which requires that the areas of the curves represented by $a_j y_j$ and $\lambda a_j \{(G,y_j) + Hy_j\}$ with the x-axis, and the first (n-1) moments of these areas about the y-axis are equal.

Another convenient set is that of the coordinate functions used in the RITZ expansion,

$$y_1, y_2, y_3, \ldots , y_n ; \tag{31}$$

and, if they are orthogonal, $\mathbf{H}$ of equation (29) reduces to a diagonal matrix whose elements are the norms of the y_j's

$$\mathbf{Da} = \lambda \mathbf{Ka} \quad \text{with} \quad \mathbf{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & d_n \end{pmatrix}; \; d_j = (y_j, y_j) = N\overline{y_j} \; . \tag{32}$$

Premultiplying both sides by the inverse of the diagonal matrix: $\mathbf{D}^{-1}$ and setting $\lambda = \frac{1}{\eta}$, there follows:

$$\mathbf{D}^{-1}\mathbf{Ka} = \eta \, \mathbf{a} \quad , \quad \text{or} \quad \mathbf{Ma} = \eta \, \mathbf{a} \tag{33}$$

where $\mathbf{M} = \mathbf{D}^{-1}\mathbf{K}$. The inverse of a diagonal matrix is simply the matrix obtained by replacing the diagonal terms by their reciprocals, and is therefore again a diagonal matrix: $\mathbf{D}^{-1} = \begin{pmatrix} 1/d_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & 1/d_n \end{pmatrix}$.

Recalling the multiplication rule of diagonal matrices with matrices (eq.10, page 6), it is seen that the matrix $\mathbf{M}$ is obtained from the matrix $\mathbf{K}$ by dividing every row vector of $\mathbf{K}$ by the corresponding element of the matrix

D . Written out in detail:

$$M = D^{-1}K = \begin{pmatrix} k_{11}/d_1 & k_{12}/d_1 & \cdots & k_{1n}/d_1 \\ k_{21}/d_2 & k_{22}/d_2 & \cdots & k_{2n}/d_2 \\ \vdots & \vdots & & \vdots \\ k_{n1}/d_n & k_{n2}/d_n & \cdots & k_{nn}/d_n \end{pmatrix}$$

The order of the characteristic values ($\eta = 1/\lambda$) is now reversed.

One could, of course, also have used an ortho-normal set,

$$(y_i , y_j) = e_{ij}$$

in the RITZ expansion and would then have arrived directly at eq. 33.

These last methods are sometimes called <u>methods of weighting functions</u>.

The methods presented so far are based upon the assumption that the integral equation (2), page 46, (which is to be solved) can be put into the form (5) in which the first derivative of the unknown function does not appear (see eq. 3'), and where the linearization $\alpha' \cong y'$ has been performed. The elimination of y' was achieved by a partial integration, and in carrying this out it was necessary to calculate the partial derivative with respect to ξ of the Green's function $H(x;\xi)$, as defined in eq. 4, pg. 47. In applications it is often difficult to perform this differentiation, and in cases where it is easier to obtain the influence functions numerically in the form of a matrix (which is merely a function-table listing the deflections at a certain point due to a unit load at various other points or the same point) one would have to perform the differentiations numerically also. These are relatively cumbersome and, as with all differentiations, represent a loss of accuracy. Thus

it is desirable to treat equation (2) directly without linearizing it and without transforming it into the form (5'), page 47. This requires a modification of the methods that have been outlined, as is shown in the section below.

e) Approximate Solutions of the non-linear Integro-Differential Equation

The equation (2) of section c) (pg. 46):

$$y(x) = \lambda \int_a^b \left\{ g(x;\xi)\mu(\xi)y(\xi) + \frac{\partial g(x;\xi)}{\partial \xi}\tau(\xi)\frac{d\alpha(\xi)}{d\xi} \right\} d\xi \quad , \qquad (2)$$

where $g(x;\xi)$, $\partial g(x;\xi)/\partial\xi$, $\mu(\xi)$, $\tau(\xi)$ are known functions, has to be solved.

Let the unknown function $y(x) = \alpha(x) + \beta(x)$ be approximated by a RITZ expansion:

$$\left.\begin{aligned} y(x) \simeq \bar{y}(x) = a_j y_j(x) = a_j\left[\alpha_j(x) + \beta_j(x)\right] \quad , \\ y_j = \alpha_j + \beta_j \quad , \end{aligned}\right\} \qquad (11)$$

where the coordinate functions $y_j(x)$ individually satisfy the boundary conditions, are linearly independent, and are preferably rough approximations to the natural modes. Take them, for instance, as equal to the deflections of the beam due to the mode-loads of a uniform beam with the same boundary conditions.

The static mode-loads of non-uniform beams, neglecting the influences of shear and rotary inertia, satisfy the equation (see eq. 4', chpt. III):

$$(E I y_k'')'' = \omega_k^2 \mu y_k \quad , \qquad k = 1,2,\ldots,n,\ldots$$

The static deflection of the same beam due to a load p(x) is given by:

$$(EIy'')'' = p$$

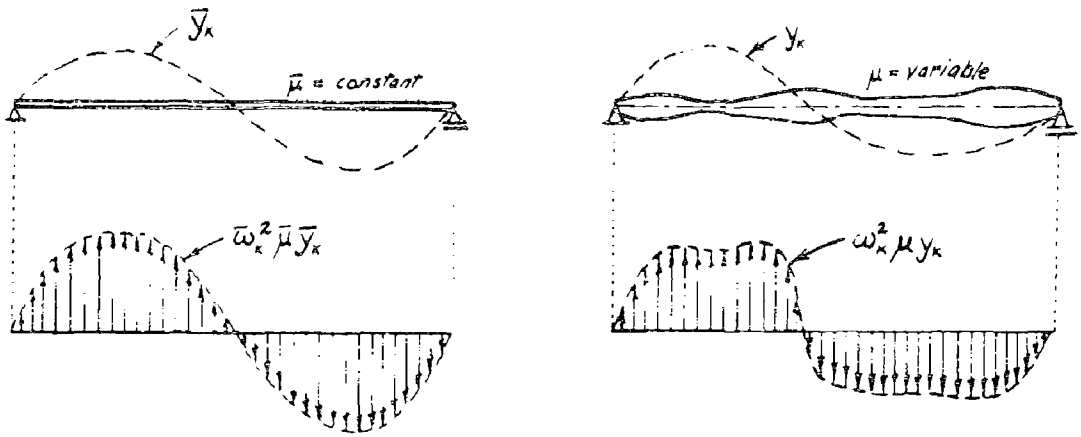
Comparing this equation with the preceding one, it is apparent that the natural modes are identical with the static deflections due to the loads

$$p_k = \omega_k^2 \mu y_k$$

for, the equality of the right hand sides of both equations necessitates the equality of their left-hand sides. The loads p_k are the so-called <u>static mode loads</u>.

It has been shown that the function $\{\sqrt{\mu}\, y_k\}$ are orthogonal. From the last relation it follows therefore that the function $\{p_k/\sqrt{\mu}\}$ are orthogonal. They can readily be normalized.

For a uniform beam with both ends supported the modes are trigonometric functions (sine waves); other support conditions yield products of trigonometric and hyperbolic functions. If the uniform beam is gradually transformed into a non-uniform beam, the modes transform (as do the natural frequencies) continuously into those of the non-uniform beam. Their shape still resembles the sine curves (or modified sine curves) of the uniform beam. The mode loads of the non-uniform beam, however, being products of the normal modes (or <u>normal functions</u> as they are also called) and the mass density distribution function μ , will look very much different from the smooth mode-loads of the uniform beam. The following picture shows on the left the (second) mode and the corresponding mode-load of a uniform beam. On the right is the corresponding mode and mode-load of a non-uniform beam with the same conditions of support.



Thus, it is indeed obvious that the product of y_k times μ (variable) resembles $\mu\, y_k$ very much, and it is concluded that the coordinate functions in the Ritz expansion (11) ought to be obtained as the static deflections due to the loads which are the product of the uniform beam modes times the mass density distribution function μ of the non-uniform beam.

The evaluation of y_j and α_j does not necessiate additional calculations, since y_j has to be obtained as the sum of α_j and β_j .

After this excursion on a rational choice of the coordinate functions of the RITZ expansion, there remains to show how equation (2) is to be solved.

Two basically different methods will be discussed as in section d), viz.: collocation - and weighting function methods. Since the underlying principles have already been established the argument need not be long.

The integral equation (2), which must be solved, is of the form:

$$y = \lambda \left\{ (K, y) + (M, \alpha') \right\} , \qquad (2')$$

wherein the abbreviations:

$$\left.\begin{aligned} (K, y) &\equiv \int_a^b g(x;\xi)\mu(\xi)\, y(\xi)\, d\xi \,, \\ (M, \alpha') &\equiv \int_a^b \frac{\partial g(x;\xi)}{\partial \xi}\, \tau(\xi)\, \frac{d\alpha(\xi)}{d\xi}\, d\xi \end{aligned}\right\} \qquad (34)$$

have been introduced, and where the definite integrals are written as inner products. Introducing the two RITZ expansions

$$y \approx \bar{y} = a_j y_j \quad ; \quad \alpha' \approx \bar{\alpha}' = a_j \alpha_j' \qquad (11)$$

into (2[4]) one obtains:

$$a_j y_j \approx \lambda\, a_j \left\{ (K, y_j) + (M, \alpha_j') \right\} \,. \qquad (35)$$

The coefficients a_j remain to be determined in such a way as to minimize the discrepancy between the left - and right hand side of this approximate equality. Since n coefficients a_j are "free", one must prescribe n conditions which have to be satisfied:

<u>collocation at n points</u> is one possible set of conditions.

<u>orthogonality to the first n members of a complete set</u> of functions is another one.

<u>Collocation</u> at n distinct points transforms the approximate equality (35) into the following system of homogeneous linear equations for the a_j's:

$$a_j y_j(x_i) = \lambda\, a_j \left\{ \left(K[x_i;\xi], y_j[\xi] \right) + \left(M[x_i;\xi], \alpha_j'[\xi] \right) \right\} \,. \qquad (36)$$

With the matrices and vector defined by:

$$\mathbf{Y} = (y_{ij}) \; ; \quad \mathbf{K} = (\kappa_{ij}) \; ; \quad \mathbf{M} = (m_{ij}) \; ; \quad \mathbf{a} = \{a_j\} \; ; \tag{37}$$

$$y_{ij} = y_j(x_i) \; ; \quad \kappa_{ij} = \left(K[x_i;\xi], y_j[\xi]\right) \; ; \quad m_{ij} = \left(M[x_i;\xi], \alpha'[\xi]\right) \; ;$$

the system (36) takes the form:

$$\mathbf{Ya} = \lambda \mathbf{Na} \tag{38}$$

where the matrix

$$\mathbf{N} = \mathbf{K} + \mathbf{M} \tag{39}$$

has been introduced. The condition that (38) possesses other than the trivial solution, $\mathbf{a} = 0$, yields the lambda equation:

$$\left\| \mathbf{Y} - \lambda \mathbf{N} \right\| = 0 \tag{40}$$

for the characteristic frequencies.

As the coordinate functions y_j and α'_j are not linearly dependent, the matrix $\mathbf{Y}$ is not singular and one can premultiply both sides of (38) by $\mathbf{Y}^{-1}$. With:

$$\eta = 1/\lambda \tag{41}$$

equation (38) appears as:

$$\mathbf{Pa} = \lambda \mathbf{a} \quad ; \quad \text{where} \quad \mathbf{P} = \mathbf{Y}^{-1}\mathbf{N} \tag{42}$$

The associated characteristic equation is:

$$\left\| \mathbf{P} - \lambda \mathbf{E} \right\| = 0 \; . \tag{43}$$

The matrix P is best obtained as solution $P = X$ of the matrix equation:

$$Y X = N \qquad X = Y^{-1} N \quad , \tag{44}$$

where the n right hand sides are the n column vectors n_j .

Weighting function method requires that both sides of eq. (35) be orthogonal to n members of a complete set of functions $\{\varphi_n\}$, viz.

$$a_j(y_i, \varphi_i) = \lambda a_j \left\{ \left((K, y_j), \varphi_i \right) + \left((M, \alpha_j), \varphi_i \right) \right\} \quad . \tag{45}$$

With:

$$\mathbf{y} = (y_{ij}) \; ; \quad y_{ij} = (y_j, \varphi_i) \; ; \quad K_{ij} = \left((K, y_j), \varphi_i \right) \; ; \quad \mathbf{K} = (K_{ij}) \; ; \tag{46}$$

$$M_{ij} = \left((M, \alpha_j), \varphi_i \right) \; ; \quad \mathbf{M} = (M_{ij}) \; ; \quad \mathbf{N} = \mathbf{K} + \mathbf{M} \; ; \quad n_{ij} = (K_{ij} + M_{ij}) \quad ,$$

this reduces to the familiar generalized eigenvalue equation:

$$\mathbf{y a} = \lambda \mathbf{N a} \tag{47}$$

with the associated frequency equation (lambda equation):

$$\| \mathbf{y} - \lambda \mathbf{N} \| = 0 \quad . \tag{48}$$

Again, $\mathbf{N}$ is not singular, hence one may also write:

$$\mathbf{P a} = \lambda \mathbf{a} \tag{49}$$

where $P = N^{-1} y$; $N X = y$, $X = P$;

with the frequency equation:

$$\left\| P - \lambda E \right\| = 0 \quad . \tag{50}$$

If the first n members of the complete set are the functions:

$$x^0,\ x^1,\ x^2,\ \ldots,\ x^{n-1},$$

it is said that one requires equality between the areas under the curves representing the left - and right hand side of (35) and between their first (n-1) moments about the y-axis.

Other suitable sets are the coordinate functions of the Ritz expansion. If these are orthogonal a reduction of labor is effected. The expansion functions which are recommended for use are orthogonal with respect to the mass distribution function μ as a weighting function. Their construction has been described in full detail on pg. 60.

Related to the weighting function method is the modified collocation method - outlined on pg. 54,55 - where it is required that the integrals over n adjacent subintervals of both sides of eq. (35) be equal. This method is, of course, less cumbersome than the weighting function method itself, as the integrations need not be carried out over the whole length of the beam, but only over n small portions of it. Numerical integration will have to be used in practically all cases, and thus one integration process will yield the eigenvalue equations, if only the cumulative results of this numerical integration are registered at the n points. For, truly, the difference between the partial sums at x_i and x_{i-1} is equal to the integral of the function over the interval x_{i-1} to x_i.

f) On the Convergence of the Collocation Method

Collocation methods have been used in the preceding sections without, however, demonstrating that they lead to results that approximate indeed the exact solution. Readers who are familiar with the Fredholm theory of integral equations will know that collocation methods are identical with the classical treatments of linear integral equations by Volterra and Fredholm. (A complete presentation of these methods, and in the author's opinion - the clearest one, is given in G. Kowalewski, Integralgleichungen, Berlin 1930.) It is instructive to outline the methods once more and to state the most important findings of the theory with respect to application.

Consider that the following (non-homogeneous) linear integral equation has to be solved:

$$f(x) - \int_a^b K(x;y)\, f(y)\, dy = g(x)$$

in the interval $a \leq x \leq b$. Let x_i be a point in the interval $[a,b]$. Then at $x = x_i$ the equation:

$$f(x_i) - \int_a^b K(x_i;y)\, f(y)\, dy = g(x_i)$$

is certainly satisfied. Replace the integral by an approximating sum:

$$\int_a^b K(x_i;y)\, f(y)\, dy \approx \sum_j^n K(x_i;y_j)\, f(y_j)\, \Delta y_j$$

and let the interval $[a,b]$ be subdivided into n equal intervals, then one obtains the following n simultaneous equations:

$$f(x_i) - \sum_j K(x_i, y_j)\, f(y_j)\, \Delta y_j = g(x_i)$$

which are satisfied only approximately by the exact solutions of the integral equation. x_i and y_j take on all values $a + k(b-a)/n$; $(k = 1,2,..,n)$: independently. With $f_i = f(x_i)\sqrt{\Delta x_i}$; $g_i = g(x_i)\sqrt{\Delta x_i}$; $K_{ij} = -K(x_i, y_j)\sqrt{\Delta x_i\, \Delta y_j}$, one arrives at the equations: $f_i + \sum_j K_{ij} f_j = g_i$ or written out in full:

$$\begin{aligned}
(1 + K_{11})\, f_1 &+ K_{12} f_2 + \cdots + K_{1n} f_n = g_1 \\
K_{21} f_1 &+ (1 + K_{22}) f_2 + \cdots + K_{2n} f_n = g_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
K_{n1} f_1 &+ K_{n2} f_2 + \cdots + (1 + K_{nn}) f_n = g_n
\end{aligned}$$

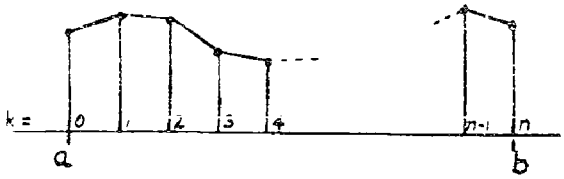
Cramer's rule gives the solutions of this system as:

$$f_r = \frac{D_r}{D} \qquad (r = 1,2, \ldots, n)$$

where D is the determinant of the coefficients of the unknowns, and D_r is obtained from D by replacing the r-th column in D by $\{g_1, g_2, \ldots, g_n\}$.

$$D = \begin{vmatrix} (1+K_{11}) & K_{12} & \cdots & K_{1n} \\ K_{21} & (1+K_{22}) & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & (1+K_{nn}) \end{vmatrix} ; \quad D_r = \begin{vmatrix} (1+K_{11}) & \cdots & K_{1,r-1} & g_1 & K_{1,r+1} & \cdots & K_{1n} \\ K_{21} & \cdots & K_{2,r-1} & g_2 & K_{2,r+1} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{n1} & \cdots & K_{n,r-1} & g_n & K_{n,r+1} & \cdots & K_{nn} \end{vmatrix} .$$

Suppose the f_k have been calculated, based on a subdivision of the interval $[a,b]$ into n equal parts, and the values of the ordinates are connected by straight lines, as the following sketch illustrates.



The question arises now as to what happens if the subdivision is made smaller and smaller.

This necessitates the investigation of the so-called Fredholm determinant of the Kernel $K(x,y)$ which is the limit of D as the number n of subdivisions of $[a,b]$ tends to infinity.

The Fredholm theory ascertains the uniform convergence of the approximative solutions towards the exact solution as the subdivisions of the interval $[a,b]$ become smaller and smaller.

This important result justifies the collocation method.

g) The Method of Station Functions (another method of constructing coordinate functions for the RITZ expansion)

Quite frequently one finds the suggestion that the so-called station function method yields suitable coordinate functions for the RITZ expansion. This recommendation will now be examined critically.

Recall that the convergence of the RITZ expansion is best, if the assumed functions are as close as possible to the actual eigenfunctions of the problem under investigation. In section e) it was suggested that the eigenfunctions of a "neighboring" problem be used to construct an approximation to the mode-loads of the actual problem, and then to calculate the coordinate functions as the static deflections under these loads. These offer the great

advantage of being orthogonal functions, thereby reducing the numerical calculations to solve the matrix eigenvalue equation. But furthermore they are already close to the actual modes. The station function method is now developed.

Suppose the linear homogeneous integral equation (Fredholm equation of the second kind) has to be solved,

$$f(x) = \lambda \int_a^b K(x,y)\, f(y)\, dy \quad , \tag{51}$$

where the kernel K(x,y) is given. As in the Fredholm theory, replace the integral equation by a set of n linear homogeneous equations. The solution to this set does not satisfy the integral equation exactly. Through the n points an interpolation function can be fitted which permits the determination of functional values at all points between a and b. Since n homogeneous linear equations admit, in general, n solutions which correspond to the n roots of the characteristic equation, one provides for this fact by writing the solution as a linear aggregate of n interpolation functions:

$$f(x) = f(x_i)\, g_i(x) \tag{52}$$

Because f(x) must take on the value $f(x_i)$ for $x = x_i$, it is apparent that the functions g(x) are subject to the requirement:

$$g_j(x_i) = e_{ji} \quad . \qquad \left[\mathbf{E} \equiv (e_{ij})\right] \tag{53}$$

The functions g_j have to satisfy the boundary conditions and should, if possible, be identical with the normal functions of the problem. It is recalled that the kernel function is an "elementary solution" of the problem

from which all solutions can be built up by means of the superposition principle. If $h_j(x)$ is a load-function, then

$$\int_a^b K(x;y)\, h_j(y)\, dy$$

is a solution of the problem which satisfies all boundary conditions. As a matter of fact, this integral represents the effect due to the cause $h_j(x)$. Since every $g_i(x)$ has to satisfy n conditions, viz.

$$g_i(x_1) = g_i(x_2) = \cdots = g_i(x_{i-1}) = 0, \quad g_i(x_i) = 1, \quad g_i(x_{i+1}) = \cdots = g_i(x_n) = 0,$$

and a total of n functions $g_i(x)$ have to be constructed, n^2 free parameters must be chosen such that all these conditions can be fulfilled. Consequently the g_i's must have the form:

$$g_i(x) = a_{ij} \int_a^b K(x;y)\, h_j(y)\, dy, \tag{54}$$

where the load functions $h_j(x)$ are given functions. They ought to be approximations to the mode-loads, under which condition the diagonal elements a_{ii} of the matrix $\mathbf{A} = (a_{ij})$ will be much larger than the other ones.

One is now left with the task of evaluating the matrix $\mathbf{A}$ so as to make $g_i(x_j) = e_{ij}$. Let $\mathbf{g}$ be the vector: $\{g_1, g_2, \cdots, g_n\}$, $\mathbf{k}$ the vector $\{k_1, k_2, \cdots, k_n\}$ with an element $k_j = \int_a^b K(x;y)\, h_j(y)\, dy$; then the equations (54) appear as:

$$\mathbf{g} = \mathbf{A}\mathbf{k} \tag{54'}$$

If x takes on the values $x_1, x_2, \ldots, x_n$ this becomes a set of n simultaneous equations. With the definitions: $g_i(x_j) = g_{ij}$, $k_j(x_i) = k_{ji}$, they are conveniently written as $\mathbf{G} = \mathbf{A}\mathbf{K}$. (55)

In view of equation (53), $\mathbf{G}$ is the unit matrix $\mathbf{E}$, and thus $\mathbf{A}$ is the reciprocal of $\mathbf{K}$:

$$\mathbf{A} = \mathbf{K}^{-1} \tag{56}$$

where $a_{ij} = (k_{ij})^{-1}$ and $k_{ij} = \int_a^b K(x_j; y)\, h_i(y)\, dy$.

(Note the order of the indices!) Once the a_{ij}'s are known, one may substitute eq. (54) into eq. (51) to obtain:

$$f(x) = \lambda \int_a^b K(x; y) \left[\sum f(x_i)\, g_i(y) \right] dy \quad , \tag{57}$$

or also:

$$f(x) = \lambda \sum_i^n f(x_i) \int_a^b K(x; y)\, g_i(y)\, dy \quad . \tag{57'}$$

This equation has to be satisfied for all x, especially for $x = x_j$ ($j = 1, \ldots, n$) and it possesses a non-trival solution only if

$$\det. \left[-\frac{1}{\lambda} f_i + \sum_i^n f_i\, k_{ij} \right] = 0 \quad ; \quad (\text{where } f(x_j) = f_j) \tag{58}$$

is satisfied. The roots of the frequency equation determine n sets of solutions $\{ f_i \}$ which, when substituted into (52) give a continuous approximation to the solutions of the integral equation.

The integrations occuring in (58) will, in general, have to be carried out numerically. Gaussian quadrature yields the highest degree of accuracy for a given number of integration stations. It will, however, seldom be applied, because the values of the function will be given at equal intervals. In any case, the number of integration stations has to be large

enough to render the error of the numerical integration insignificant compared with the approximation introduced with (52).

It is apparent that the station function method is very laborious. In the present problem its application is not justified, because exact solutions of neighboring problems can be found readily and be used in a RITZ expansion. The merits of the station function method lie in flutter analysis of aircraft wings and not in the vibration analysis of beams. There the method, extended to two independent variables (planar problems) is really useful, for usually only the lowest frequency is of interest and coordinate functions to be utilized in a RITZ expansion are not easy to find.

4. Limitation of the Analytical Methods

In section 1) of this chapter five restrictions have been found that must be imposed upon the vibrating beam in order that the present analysis be applicable and it had been stated that other restrictions, of dynamical nature, would be formulated later. To these attention is now turned. Recall briefly the "integral methods" that have been outlined, viz. the variational method and the integral equation method. The first of these states that the total energy of the vibrating body must be a minimum for every natural mode. The second states that the natural modes are the characteristic solutions of an integral equation which represents all possible vibrations of the body. The unknown solutions are written as linear aggregates of known vibrations and the problem immediately arises of determining the degree of participation of these known vibrations in order to build up the desired solutions. This leads then to an algebraic problem of solving a matrix eigenvalue equation. Every solution vector corresponds to characteristic solution of the problem. One

would be tempted to believe that any desired degree of accuracy can be obtained if only the Ritz expansion (i.e. the linear aggregate) comprised a sufficiently large number of linearly independent functions and corresponding participation factors. But this is not so! Suppose that a Ritz expansion contains twenty members and that the eigenvalue problem has been solved. Then one may write down the expansion of the, say, tenth natural mode of the beam and one knows from mathematical reasoning that this ought to be a very good approximation to this mode, if the linear aggregate represents the solutions of a neighboring problem. Let the beam under investigation have a length of ten times its average height. Under these assumptions a half-wave of the tenth mode has a length of about one tenth of the span of the beam. Therefore, the nodal points subdivide the beam into sections with a length to height ratio of approximately one to one. Between two adjacent nodal points the beam deflects as if it were a portion of a continuous beam simply supported at the nodal points. The simplified beam theory is, however, not applicable to such stubby beams, since the stress-strain distribution across the beam is no longer linear. Shear deformation becomes so great that the warping of the cross sections can no longer be neglected. Furthermore the normal stresses perpendicular to the beam axis must be considered. This means that a solution satisfying the rigor of the mathematical theory of elasticity is called for. Unfortunately no such vibration solutions have been found and it is not possible to compare the engineering solution with the elasticity solution directly. Bending problems, however, have been solved in great number and there it was found that as long as the beam has a length of more than three times its average depths, the engineering formulas give a reasonable approximation to the

behavior of the beam. The factor three cannot be taken religiously, for it depends on the boundary conditions, the geometry of the beam, and the degree of accuracy required. It is reasonable to assume that the vibration analysis obeys the same general restrictions as the static deflection analysis and, until more constraining requirements can be derived from exact analysis, it is recommended that only those modes should be calculated by the methods developed in this report which yield half-wave lengths that are long in comparison to the average depth of the beam (i.e. at least three to five times the depth).

The restriction just voiced should not mislead the analyst to the conclusion that is is useless to carry many terms in a RITZ expansion, believing that the higher terms represent the higher modes of the beam. This is not true, for, if one had an analytical expression in closed form, say of the first mode, and expanded this into a uniformly converging series and used the expansion instead of the closed form in the calculations, then certainly all terms of the expansion are needed to represent the mode properly. The limitation of the method is of a different nature. It states only that the lower modes can be calculated by the formulas given, because the equations are valid only in a domain of low frequencies. Higher frequencies and modes do exist but they cannot be calculated with a vibration theory that is based upon a simplified beam analysis.

Furthermore, it must be emphasized that the beams to be investigated have to possess a symmetric cross section and the vibrating has to occur in the plane of symmetry. Eccentric, fixed loads on the beam are thus to be excluded, as no coupling of bending - and torsional vibrations is permissible.

Only recently R. Heilig, Ingenieur- Archiv, vol. 19, pg. 231-254, (1951), has developed a rigorous solution for the coupled torsion-bending vibration of prismatic, thin-walled, open-profile beams, taking proper account of the warping stresses. An extension of this theory to prismatic bars of arbitrary cross section encounters great difficulties which arise from the fact that, at present, it is not possible to calculate the stress distribution in a prismatic bar of arbitrary cross section, with arbitrary boundary conditions, due to bending and torsion. Efforts ought to be made in this direction, and the above mentioned paper gives a good starting ground.

The findings of this report are summarized in the following:

Lemma: The methods presented in this report are applicable to straight beams with symmetric cross section which vibrate in the plane of symmetry. They yield satisfactory results for modes whose half-wave length is long in comparison to the average height of the beam.

384 VII. Tabellen.

$\frac{x}{l}$	$\lambda = 1.875104$ $\lambda^2 = 3.516015$		$\lambda = 4.694091$ $\lambda^2 = 22.03449$		$\lambda = 7.854757$ $\lambda^2 = 61.697207$		$\lambda = 10.995541$ $\lambda^2 = 120.901922$		$\lambda = 14.137168$ $\lambda^2 = 199.859519$	
	Y_1/l	$Y_1'l$	Y_2/l	$Y_2'l$	Y_3/l	$Y_3'l$	Y_4/l	$Y_4'l$	Y_5/l	$Y_5'l$
0,00	0,00000	7,0321	0,00000	64,069	0,00000	123,394	0,00000	241,81	0,00000	399,72
0,05	0,00840	6,5381	0,07215	33,067	0,13407	75,082	0,24694	+ 109,67	0,38224	+ 120,75
0,10	0,03334	6,0044	0,18130	23,168	0,45599	+ 28,187	0,77001	12,59	1,07449	- 117,52
0,15	0,07518	5,3472	0,37757	12,853	0,84960	- 14,292	1,26750	- 106,79	1,49511	- 250,86
0,20	0,12774	5,1016	0,60211	+ 3,086	1,20901	- 48,725	1,50758	- 155,49	1,31925	- 240,01
0,25	0,19458	4,6253	0,83452	- 5,949	1,44899	- 71,748	1,37031	- 150,26	+ 0,57040	- 102,33
0,30	0,27297	4,1550	1,05237	- 13,975	1,51246	- 81,123	0,85500	- 94,55	- 0,42257	+ 90,23
0,35	0,36175	3,6933	1,23536	- 20,722	1,37613	- 76,264	+ 0,13148	- 10,93	- 1,19863	242,44
0,40	0,45976	3,2427	1,36694	- 25,978	1,05185	- 58,445	- 0,63112	+ 78,94	- 1,39312	279,90
0,45	0,56590	2,8063	1,43393	- 29,580	0,58349	- 30,761	- 1,20110	146,36	- 0,91632	183,99
0,50	0,67905	2,3875	1,42733	- 31,450	+ 0,03937	+ 2,430	- 1,41424	170,98	+ 0,00170	+ 0,34
0,55	0,79817	1,9897	1,34247	- 31,596	- 0,49858	35,999	- 1,21057	145,21	0,92062	- 183,13
0,60	0,92227	1,6165	1,17895	- 30,120	- 0,94752	64,896	- 0,65300	+ 76,30	1,40033	- 278,02
0,65	1,05042	1,2719	0,94044	- 27,221	- 1,23611	85,654	+ 0,09043	- 15,90	1,21303	- 235,56
0,70	1,18175	0,9598	0,63421	- 23,188	- 1,31486	93,314	0,78204	- 103,37	+ 0,45146	- 86,46
0,75	1,31549	0,6841	+ 0,26997	- 18,388	- 1,15289	89,399	1,24285	- 165,68	0,51200	+ 116,00
0,80	1,45096	0,4492	- 0,14006	- 13,267	- 0,78974	74,592	1,28610	- 182,27	- 1,20091	263,67
0,85	1,58907	0,2643	- 0,58331	- 8,320	- 0,23166	52,419	0,88330	- 153,25	- 1,25519	298,81
0,90	1,72681	0,1180	- 1,05146	- 3,995	+ 0,45687	28,133	+ 0,10406	- 93,10	- 0,58801	214,73
0,95	1,86238	0,0303	- 1,50470	- 1,490	1,21695	8,273	- 0,90719	- 29,85	+ 0,60416	76,65
1,00	2,00000	0,0000	- 2,00000	- 0,000	2,00000	0,000	- 2,00000	- 0,00	2,00000	0,00

THE FIRST FIVE NORMAL MODES OF A UNIFORM CLAMPED-FREE BEAM

(Reproduced from Dynamik der Stabwerke by Z. Hohenemser and W. Prager, Springer, 1933)

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